Comparison 12: Collision Processes in Rows of Spheres

This comparison deals with a model of the mechanics. The features to be compared represent a large number of events, the numerical accuracy, the iteration of a boundary value, and stochastic parameter variations. Piecewise, constant velocities permit both a continuous and a discrete treatment.

Subject of the investigation are sequences of collisions, caused by the impact of a sphere on a resting row of spheres. In the elastic case only one impact occurs between neighbouring spheres, whereas one can observe many interactions if elasticity decreases. Numerical problems result from the peculiarity, that the relative distances and velocities at a low elasticity can be smaller by orders of magnitude than the absolute variables. In order to avoid small faulty differences of great values, the relative quantities are used as variables, and absolute quantities are obtained by summation.

Partially elastic collision of two masses

The collision shall take place at $t = 0$ with the velocities v_1, v_2 (Figure 1a). The force $F(t)$ being exerted from both masses on each other, rises first with t and reaches its maximum at $t = t^*$ (Figure 1b). In this compression phase, the bodies are increasingly deformed in the immediate vicinity of the contact place. At the end (maximum deformation) both bodies have the same velocity v^*. In the following restitution period the deformations disappear partially

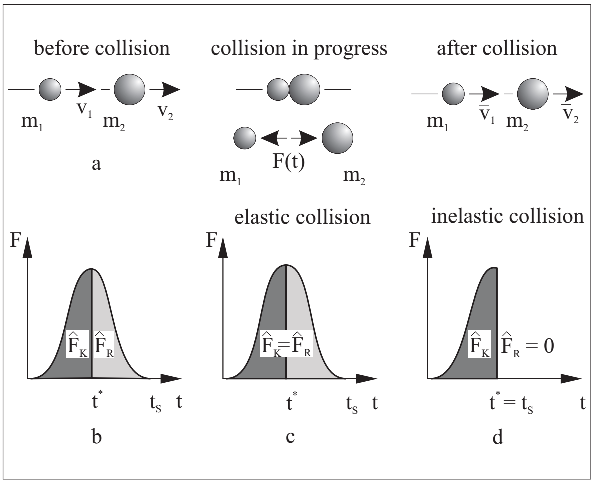

Figure 1: Central impact of two masses

or completely, concurring with a reduction of the contact force $F(t)$. After the time interval t_S the collision process is finished and both masses move with velocities $\bar{v}_1$ and $\bar{v}_2$, respectively.

The force impulses $\hat{F}_K$ and $\hat{F}_R$, exerted during both periods, determine the momentum change. They are represented by the areas below the force curve $F(t)$. The force impulse in the restitution phase reaches at most the value of the compression phase:

$$\hat{F}_R = e\hat{F}_K \text{, with } 0 \le e \le 1 \text{ .} \qquad (1)$$

e restitution coefficient (collision coefficient)

An elastic impact has the collision coefficient $e = 1$, whereas an inelastic collision is known to have no restitution phase ($e = 0$). In general, partially elastic case the collision coefficient takes on values of $0 < e < 1$. Using the momentum conservation law, the new velocities in the next period of time follow this piecewise description:

$$\bar{v}_1 = v_1 - (1+e)\frac{m_2}{m_1 + m_2}(v_1 - v_2)$$
$$\bar{v}_2 = v_2 + (1+e)\frac{m_1}{m_1 + m_2}(v_1 - v_2) \qquad (2)$$

After the limiting process of the collision time $t_S \to 0$, the impact shall be modelled in the following as a state event that takes place immediately.

Mathematical model of a spheres row

In order to obtain an ideal translation, the p spheres arranged in a row are tied up with infinite long threads without any friction (Figure 2). The model consists of $p = 4$ spheres;

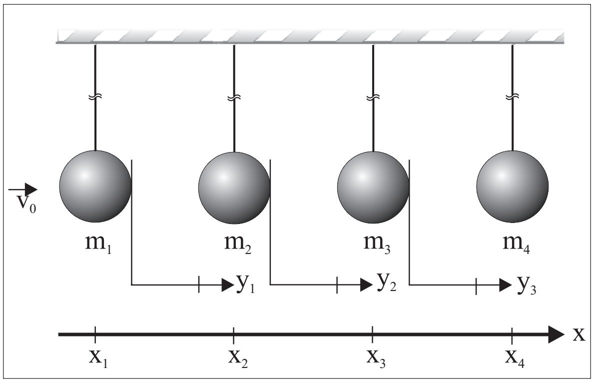

Figure 2: Collision pendulum of four spheres

for all collisions e takes on a constant value that does not depend on the velocities. A further precondition is the equality of the diameters d for all spheres, their masses m_i and distances a from each other.

In the model description the relative quantities are variables:

$$y_1 = x_2 - x_1 - d,\ y_2 = x_3 - x_2 - d,\ y_3 = x_4 - x_3 - d,$$

$$\dot{y}_1 = \dot{x}_2 - \dot{x}_1, \dot{y}_2 = \dot{x}_3 - \dot{x}_2, \dot{y}_3 = \dot{x}_4 - \dot{x}_3. \qquad (3)$$

For determination of the remaining absolute quantities by summation equations of motion for the inner distances y_i (i=1,2,3) and the absolute variable x_1 are needed. The initial conditions are chosen so that sphere 1 strikes the motionless other three spheres with velocity v_0. An influence of external forces is not considered.

Equations of motion

$$\ddot{x}_1 = 0 \quad \dot{x}_1(0) = v_0, \quad x_1(0) = 0,$$

$$\ddot{y}_1 = 0 \quad \dot{y}_1(0) = -v_0, \quad y_1(0) = a,$$

$$\ddot{y}_2 = 0 \quad \dot{y}_2(0) = 0, \quad y_2(0) = a,$$

$$\ddot{y}_3 = 0 \quad \dot{y}_3(0) = 0, \quad y_3(0) = a \qquad (4)$$

Absolute quantities

$$x_2 = x_1 + y_1 + d, x_3 = x_2 + y_2 + d, x_4 = x_3 + y_3 + d$$

$$\dot{x}_2 = \dot{x}_1 + \dot{y}_1, \dot{x}_3 = \dot{x}_2 + \dot{y}_2, \dot{x}_4 = \dot{x}_3 + \dot{y}_3 \qquad (5)$$

The expressions on the right side of equations (6) describing the velocities after a collision contain the relative velocities at the moment of impact as derivatives of the distance variables y_i, that determine the time of collision.

Collision 1-2

$$\dot{x}_1 = \dot{x}_1 + (1+e)\cdot m_2 / (m_1 + m_2)\cdot \dot{y}_1$$

$$\dot{y}_2 = \dot{y}_2 + (1+e)\cdot m_1 / (m_1 + m_2)\cdot \dot{y}_1 \qquad (6a)$$

$$\dot{y}_1 = -e \cdot \dot{y}_1$$

Collision 2-3

$$\dot{y}_1 = \dot{y}_1 + (1+e)\cdot m_3 / (m_2 + m_3)\cdot \dot{y}_2$$

$$\dot{y}_3 = \dot{y}_3 + (1+e)\cdot m_2 / (m_2 + m_3)\cdot \dot{y}_2 \qquad (6b)$$

$$\dot{y}_2 = -e \cdot \dot{y}_2$$

Collision 3-4

$$\dot{y}_2 = \dot{y}_2 + (1+e)\cdot m_4 / (m_3 + m_4)\cdot \dot{y}_3 \qquad (6c)$$

$$\dot{y}_3 = -e \cdot \dot{y}_3$$

Insignificant or positive relative velocities $(\dot{y}_1 \geq 0) \wedge (\dot{y}_2 \geq 0) \wedge (\dot{y}_3 \geq 0)$, i.e., monotonously increasing absolute velocities, establish the termination criterion for a simulation run, that is, no further collisions will occur and the velocities will not change.

Tasks

Task a)

a1) Graphical representation of the distance-time functions $y_1(t)$, $y_2(t)$ and $y_3(t)$ for parameter values $e = 0.2$, $d = 1$ and initial values $a = 1$, $v_0 = 1$ in time interval $0 \leq t \leq 15$ (termination criterion met). Initial values and sphere diameter d remain valid in the following.

a2) Final values of the velocities for $e = 1$ (elastic case) and for the quasi-plastic case in which velocities are sufficiently equal.

Task b)

b1) Number of collisions as a function of the restitution coefficient $n(e)$ which should be varied from $e = 1$ to a value for which the quasi-plastic case is reached.

b2) Graphical representation of the final velocities $\dot{x}_1$, $\dot{x}_2$, $\dot{x}_3$ and $\dot{x}_4$ as a function of values of e for $e \leq 1$ up to the quasi-plastic case.

Task c)

c1) As a boundary value problem the restitution coefficient e is to be determined such that the final velocity be $v_4 = v_0/2$.

c2) The restitution coefficient e, which is equal for all spheres, is now a normally distributed stochastic variate with mean value $m = 0{,}5$ and standard deviation $s = 0{,}05$. The distribution function of v_4, mean value, standard deviation and confidence interval with confidence probability of 95% for a sufficiently large sample size are to be determined.

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