Simulation and Optimization Framework for Stochastic Resource-Constrained Project Scheduling Problems with Continuous Random Variables

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Abstract. We introduce a simulation and optimization framework for stochastic resource-constrained project scheduling problems. The stochasticity is represented by modeling job durations as continuous random variables without time discretization, while fluctuations are captured by simulating a sufficiently large number of realizations. The objective is to analyze the relationship between variability in the input parameters and the resulting makespan in order to enable a priori estimates. Estimation approaches derived from simulations of small instances can then be extrapolated to problems involving a larger number of jobs or more complex characteristics.

Introduction

Resource-constrained project scheduling problems (RCPSP) are of widespread relevance. From a theoretical perspective, they serve as research objects for combinatorial optimization, the development of heuristics and event-discrete simulation techniques. From the practical perspective, they arise in manufacturing contexts, worker allocation schemes, medical operation and surgery planning and even in packing problems – just to mention a few. In many real-world problems, process parameters and other quantities tend to fluctuate or are only known up to certain uncertainties. Consequently, improved modeling means incorporating such aspects.

We address this by considering parameters as random variables of a given distribution instead of fixed numbers.

A review of the literature underlines the importance of RCPSPs: early attempts go back to [1, 2, 3] and recent surveys addressing the variants of such problems are for example [4, 5]. In general, heuristics and the genetic algorithm play an important role [6, 7, 8], some further state-of-the-art algorithms can be found in [9, 10, 11, 12, 13]. The works [13, 14, 15, 16] pay attention to stochastic RCPSP mainly with discrete random variables.

However, the NP-hardness of RCPSP impedes finding the exact optimum of larger problem instances within a reasonable time.

Consequently, we are therefore restricted to optimizing sufficiently small problem instances. We then extrapolate the obtained findings.

Our approach focuses on continuous random variables because many relevant processes or phenomena can be modeled by normally distributed, Gamma or Beta distributed quantities.

A stochastic RCPSP is characterized firstly by the dependencies of its inner parts (called jobs), secondly by the duration of the jobs (continuous random variables) and thirdly by the number of available resources. The aim is to minimize the makespan.

Basically, we are interested in some aspects of the systematics with special attention to stochastic influences:

- How are the distributions of the input parameters and the distribution of the objective function related?

- Regarding the reliability, is it possible to estimate the variation of the objective function using the variation of the input parameters?

To address these questions, we established a four-part simulation and analysis framework:

1. Random Number Generator: provides an appropriate set of random numbers of a prescribed distribution with expectation value and standard deviation as parameters to encode the duration of all jobs.
2. Structure Analyzer: investigates the dependencies of the jobs to find critical paths (dependencies are encoded in a directed network graph).
3. Optimizer: executes the optimization of the makespan for all stochastic realizations of the RCPSP.
4. Output Analyzer: evaluates the resulting sample of realizations w. r. t. statistical characteristic values and distribution function fits.

Our paper is organized as follows: Section 1 provides the methodical basis in terms of a precise definition of the problem class scope and short notes on the Gamma and the Beta distribution while also describing the simulation procedure. Section 2 is dedicated to selected results. We summarize in Section 3.

1 Model Setup and Simulation Approach

In this section, we explain the employed methods. First, we precisely define the scope of a stochastic RCPSP. Second, we present the framework for solving such RCPSPs. We supplement these two main part with short notes on the Gamma and the Beta distribution.

1.1 Stochastic RCPSP

To focus on the underlying principle, we refrain here from features like multiple projects, multiple modes as well as transfer times or type representatives. A stochastic RCPSP is therefore described by:

- jobs $j = 1...J$ and their respective durations $d_j \in \mathbb{R}_{\geq 0}$ considered as continuous random variables; each job starts exactly once and must not be interrupted
- successor matrix $S \in \{0,1\}^{JxJ}$ defined by

$$S_{j_1 j_2} = \begin{cases} 1 & \text{if job } j_2 \text{ follows after } j_1 \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

- number of renewable resources $R \in \mathbb{N} \cup \{\infty\}$, where each job occupies exactly one resource during its execution and $R = \infty$ refers to the unconstrained problem

In summary, a stochastic RCPSP is a tuple $(J, R, (d_j), (S_{j_1 j_2}))$ supplemented by the stochastic parameters entering the vector (d_j).

1.2 Simulation and Optimization Framework

The framework consists of the following four parts.

Random Number Generator:
To simulate stochastic RCPSPs, we generated a sufficiently large sample of realizations. For each stochastic RCPSP and each job, we keep the expected value μ_j ($j = 1...J$) of the duration d_j fixed. Furthermore, the coefficient of variation η is the same for all jobs, such that the standard deviation σ_j of the duration d_j follows from $\sigma_j = \eta \cdot \mu_j$.

In addition, we choose the type of distribution (Gamma or Beta distribution, we do not consider normally distributed random variables since they may take negative values). The sample size varies from 10,000 up to one million.

Structure Analyzer:
Dealing with continuous random variables precludes simple linear optimization models with binary decision variables (see [17, 18] for instance). Since we refrain here from a time discretization, we choose the following structural approach to incorporate the limit on the number of resources.

Let the successor matrix S be given. As a first step, the tool evaluates the potentially maximal number $R_{\max}(S)$ of resources needed such that there would be no queue (and the problem is unconstrained independently of d_j). Typically, $R_{\max}(S) > R$. As a second step, the tool collects all those successor matrices $\tilde{S}$ with $\tilde{S}_{j_1 j_2} \geq S_{j_1 j_2}$ for all j_1, j_2 and $R_{\max}(\tilde{S}) = R$. Considering $\tilde{S}$ instead of S, the RCPSP becomes unconstrained.

In summary, the idea is to solve rapidly (many) unconstrained PSPs to find the optimum of the original RCPSP. That is the key point because the main issue lies in the efficient selection of all relevant $\tilde{S}$ matrices. To handle this challenge, we strongly employ the close connection between successor matrices and posets and the knowledge of such sets (generation, isomorphism classes, structural dependencies, [19, 20, 21, 22, 23]).

Optimizer:
The task of minimizing the makespan is equivalent to finding the shortest path in a weighted directed graph (encoded in $\tilde{S}$). To this end, we employ both Gurobi and the Python package Networkx to go through the list of the relevant $\tilde{S}$ matrices. We perform this evaluation for all stochastic realizations.

Output Analyzer:
This analyzing tool processes the sample of all cycle times and computes statistical quantities, for example mean value, standard deviation, coefficient of variation, skewness. In addition, it fits the data to a given type of distribution depending on multiple parameters. Particularly interesting is the relation in terms of appropriate parametrizations between input and output quantities.

All in all, the framework is completely automatized and can be combined with a simulation framework for heuristics [24, 25, 26] and with an AI tool for finding promising fit parametrizations for the Output Analyzer. However, the results of the following section focus on stochastic fluctuations.

1.3 The Gamma and the Beta Distribution

We briefly summarize some well-known properties of these two distributions both quite convenient for modeling process times or human working durations.

Let X denote a random variable with its expectation value μ, its standard deviation σ and the dimensionless coefficient of variation $\eta = \frac{\sigma}{\mu}$. The probability density function of the Gamma distribution reads as

$$f^{(\Gamma)}_{\alpha,\beta}(x) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x} \tag{2}$$

with $x \geq 0$ (arbitrarily large values are admissible), $\alpha = \frac{\mu^2}{\sigma^2}$ and $\beta = \frac{\mu}{\sigma^2}$; analogously, we have

$$f^{(B)}_{\alpha,\beta}(x) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1} \tag{3}$$

for a Beta distribution with x ranging in [0,1] (linearly scalable to the interval $[x_{\min}, x_{\max}]$),

$$\alpha = \mu\left(\frac{\mu}{\sigma^2}(1-\mu) - 1\right) \tag{4}$$

and

$$\beta = \frac{\mu}{\sigma^2}(1-\mu)^2 + \mu - 1. \tag{5}$$

The cumulative distribution functions are denoted by $F^{(\Gamma)}_{\alpha,\beta}$ and $F^{(B)}_{\alpha,\beta}$, respectively.

2 Selected Results

We picked three instances for our case study: 20 and 50 jobs for the unconstrained PSP and 10 jobs for investigating the RCPSP.

As mentioned above, a universal coefficient of variation η is fixed (see below) and the expectation value μ_j of the duration d_j is chosen randomly in the interval [0.3, 0.7].

A key result of our investigation is that we can describe the distribution of the objective function with high accuracy by a product ansatz having only two factors and four parameters even though the problem class contains many parameters.

In addition, this statement remains true for all values of the resource number R. Note that it is not sufficient to fit a simple Gamma (Beta) distribution if the input parameters are Gamma (Beta) distributed. Indeed, the second factor in the product is necessary.

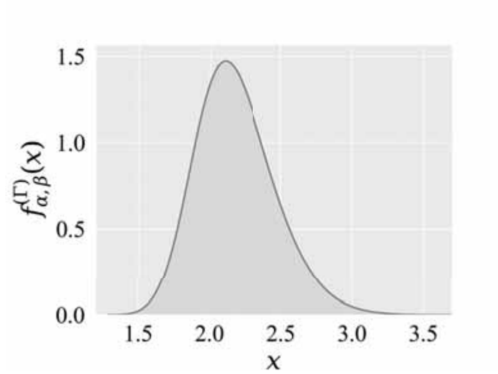

Figure 1: Histogram (blue) of the makespan x and best fits (red) according to (6) for instance with $J = 20$, Gamma distribution and $\eta = 0.3$ (sample size one million).

Let us take a closer look at the product ansatz.
We chose

$$F^{(D)}_{\alpha_1,\alpha_2,\beta_1,\beta_2}(x) = F^{(D)}_{\alpha_1,\beta_1}(x) \cdot F^{(D)}_{\alpha_2,\beta_2}(x) \tag{6}$$

with $D \in \{\Gamma, B\}$ which is inspired by the following facts:

- The sum of multiple Gamma distributed, independent random variables is Gamma distributed.

- The sum of multiple Beta distributed, independent random variables is approximately Beta distributed.
- The cumulative distribution function of the maximum of multiple, independent random variables is equivalent to the product of the cumulative distribution functions of the single random variables.

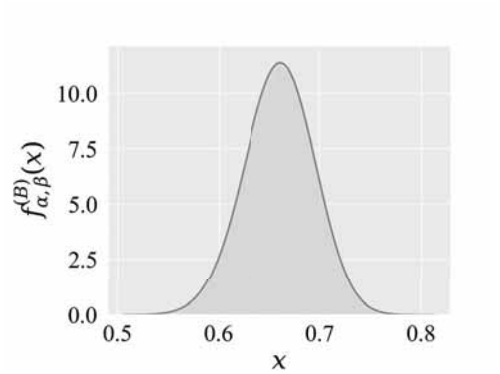

Figure 2: Histogram (blue) and best fits (red) according to (6) for instance with $J = 20$, Beta distribution, $\eta = 0.1$; the variable x is scaled to [0,1] by dividing the makespan by the longest path (sample size one million).

Here, the statements of the first two bullet points refer to the stochastic behavior along chains of jobs while the statement of the third bullet point refers to the comparison of such chains to select the critical path. However, this inspiration does not pay very much attention to the mathematical subtleties (especially the stochastic independence of the random variables).

2.1 Results of the Unconstrained PSP

We consider four samples of unconstrained problems: $J = 20$ with Gamma distribution (see Figure 1) and Beta distribution (see Figure 2), $J = 50$ with Gamma distribution (Figure 3) and Beta distribution (Figure 4). All figures display the histogram of the sample (size one million) and the best approximation according to equation (6).

The four-parameter fit exhibits a sufficiently high quality for all considered problems even though the problem class depends on far more parameters. Regarding the 50-job instances, the distribution becomes more symmetric.

As a next step, we compare dimensionless quantities. Figure 5 shows the coefficient of variation η_{out} of the objective function as a function of the coefficient of variation η_{in} of the input. Over a large number of instances, we see a stable parameter dependency.

The behavior η_{out} vs. η_{in} is close to a power law

$$\eta_{\text{out}} = a \cdot \eta_{\text{in}}^b \tag{7}$$

with $a = 0.3227 \pm 7 \cdot 10^{-4}$ and $b = 0.7874 \pm 3 \cdot 10^{-3}$ (red curve in Figure 5).

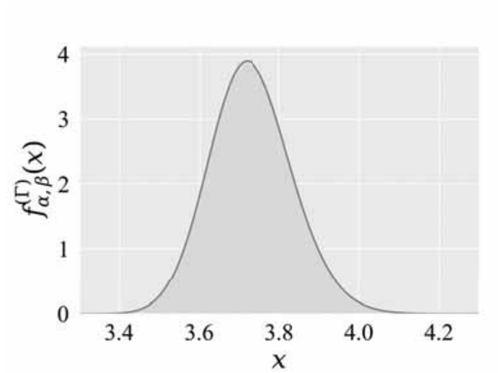

Figure 3: Histogram (blue) of the makespan x and best fits (red) according to (6) for instance with $J = 50$, Gamma distribution, $\eta = 0.3$ (sample size one million).

In addition, the objective function varies less than the input quantities. This stability in combination with the appropriate fit (6) allows an a priori estimation because the fit parameters of (6) can be estimated by a structural analysis (part of the Structure Analyzer) based on what is called the grade distribution [19, 20].

Such an estimation can be refined by a small sample. Therefore, after less runs the simulation gives already insights of the stochastic behavior of the objective function.

2.2 Results of the RCPSP

Let us now turn to the 10-job RCPSP. Mainly we are interested in two aspects:
First, does the fit ansatz (6) remain appropriate?
Second, what dependencies exist between the resource bound R and the stochastic quantities μ_{out} and η_{out}?

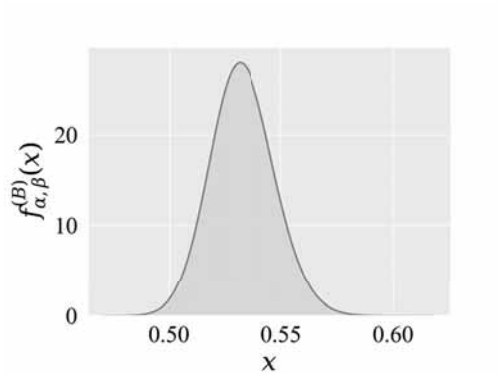

Figure 4: Histogram (blue) and best fits (red) according to (6) for instance with $J = 50$, Beta distribution, $\eta = 0.1$; x is scaled to [0,1] (sample size one million).

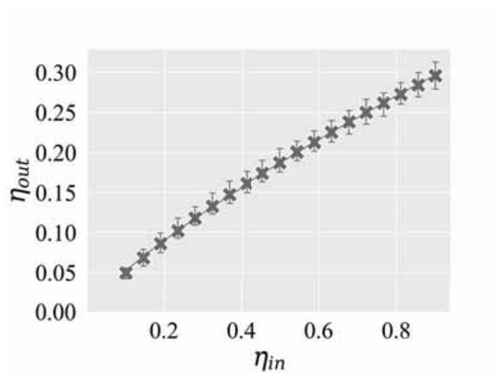

Figure 5: Coefficient of variation of the objective function η_{out} as a function of the coefficient of the variation η_{in} of the input distributions ($J = 20$, Gamma distribution, 100 instances per η_{in} value, sample size 10,000). The errorbars indicate minimum, mean (crosses) and maximum. The red curve depicts the power law regression.

Figure 6 shows the distribution (density function) of the objective function for a 10 job Gamma distributed sample for various values of R.

Note that if $R \geq R_{\max}(S)$ the problem is unconstrained (independently of R). Regarding the first question, the answer is yes (such that the histograms are not shown in Figure 6), equation (6) covers all of the following cases: for the case $R = 1$, the execution of the jobs is equivalent to a single chain. So in this special case, the second factor in equation (6) becomes obsolete.

For small values of R, the problem instance remains constrained for all possible realizations. For further increasing R, the resultant distribution contains a mixture of realizations, where the durations are such that the problem is de facto unconstrained, and realizations, where the resource constraint is indeed active.

For $R \geq R_{\max}(S)$, the problem is unconstrained for all duration vectors (d_j). Typically, $R_{\max}(S) \leq J$.

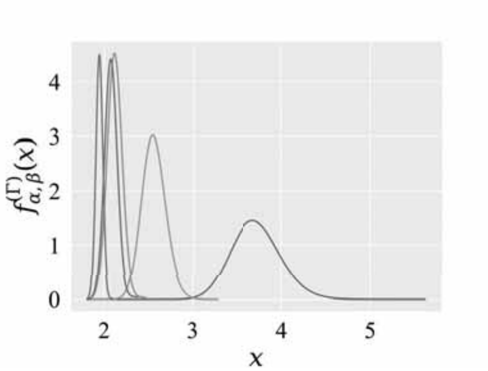

Figure 6: Distribution (density function) of the makespan for varying resource number: $R = 1$ (blue), 2 (orange), 3 (green), 4 (red) and $5 = R_{\max}$ (purple).
All cases match the fit (6). The sample size is 40,000 and $J = 10$ with Gamma distribution.

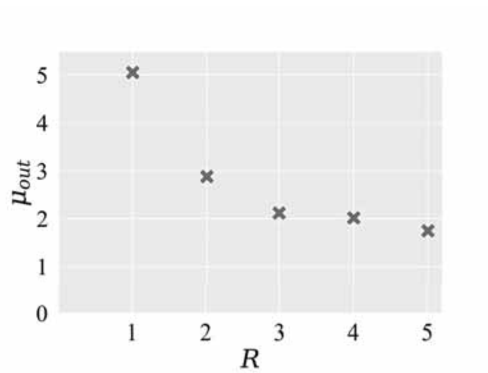

Figure 7: Dependence of expectation value μ_{out} on the resource number for a pool of Gamma and Beta distributed instances with varying μ. We have $J = 10$ and $R_{\max} = 5$ fixed such that the problem is not subject to resource limitations for $R \geq 5$.
The errorbars are not shown because they are negligible compared to the size of the crosses.

Clearly, a decreasing R causes an increasing μ_{out}. Figures 7 and 8 provide a more detailed and quantitative picture by showing μ_{out} and η_{out} as a function of R for several instances. As it is the case in Figure 5, the dependency can be captured by a rather simple regression.

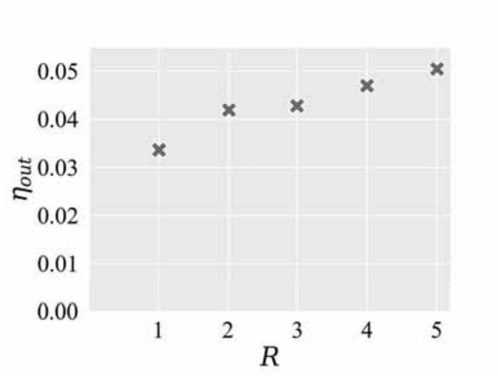

Figure 8: Dependence of the coefficient of variation η_{out} on the resource number for a pool of Gamma and Beta distributed instances with varying μ. We have $J = 10$ and $R_{\max} = 5$ fixed such that the problems is not subject to resource limitations for $R \geq 5$. The errorbars are not shown because they are negligible compering the size of the crosses.

Summarizing the previous examples, the stochastic characteristic values and the shape of the distribution of the makespan as objective function are predictable both for unconstrained and constrained PSP.

3 Summary and Outlook

The purely continuous treatment reveals close connections between the input distributions and the distribution of the makespan as objective function. Based on simulation studies, we can estimate the shape and all related stochastic properties of the resultant distribution since we found that the proposed product fit suffices for all practical purposes even though it seems hard to prove its validity by rigorous mathematical arguments.

Although solving a single realization of an RCPSP remains challenging for a larger number of jobs, considering a population significantly relaxes the situation, smooths discontinuous aspects of the combinatorial optimization problem and eventually enables statements about confidence intervals of the makespan having practical relevance.

Without aiming to explore the problem landscape too far, it seems natural to attack more sophisticated RCPSP (multi-mode and with transfer times and type representatives, for instance) with the continuous approach and to investigate the influence of these extended features on the shape of the objective function distribution.

Hereby, questions of continuous dependency and stability are of particular interest. Innovative tools in event-discrete simulation and optimization will provide a valuable contribution to this.

Publication Remark

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