Simulation of Fluid Dynamics in a Network of Blood Vessels with 1D FEM

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Abstract. The aim of this paper is to simulate the bloodstream through a network of blood vessels with a Finite Element Method in one dimension. A one dimensional system of partial differential equations is used. This system can be written in hyperbolic conservation form with the state variables cross-sectional area, the flow, the velocity and the pressure. To solve the system of partial differential equations, numerically correct boundary conditions have to be considered. For the input, a pressure function is used. To simulate the load downstream and the compliance of the arterial segments, a Windkessel model consisting of three elements is used. By simulating bifurcations the considered abstract vascular network can be build up. For that a nonlinear system of equations is set up and solved. The partial differential equation system cannot be solved analytically. Hence, to solve it a numerical Finite Element Method is used. In this context, a Taylor Galerkin method of second order with basic functions of first order is used. The model is implemented by using the mathematical software MATLAB. To verify the model, several simulations are done, using an abstract arterial tree built up by thirteen central arterial segments. In all simulations, the parameters of the Windkessel model and the parameters of the arterial segments are based on experiments and on physiological values. In all tests, physiologically realistic results are obtained.

It can be concluded that the application of a one dimensional Finite Element Method approach along with the particular implementation presented can describe the effects in a system of human arteries in a realistic way.

Introduction

Cardiovascular diseases are the most common cause of death in the modern society. To improve the diagnosis and further on the therapy of such disease, more often dynamic models for the heart circulation system are used. In these models the main factors which must be considered are accurateness, computing time and identifiability of the parameters. Therefore one dimensional models, which have in fact a hight efficiency, come to the centre of attention.

1 Methods & Models

1.1 The model

A one dimensional model of the following form is used

$$\frac{dA}{dt} + \frac{dQ}{dz} = 0$$
$$\frac{dQ}{dt} + \alpha \frac{d}{dz}\frac{Q^2}{A} + \frac{A}{\rho}\frac{dP}{dz} + K_R \frac{Q}{A} = 0 \tag{1}$$

In this system of partial differential equations the calculated parameters will be the area of the arterial segments A, the pressure P and the flow Q.

The other occurring values are set to be constant and are namely the blood density ρ, the Coriolis coefficient α, which defines the velocity profile, and a friction parameter which is given through

$$K_R = 8\pi\nu \qquad (2)$$

where ν is the viscosity of blood. In equation (1), one has two equations for the three parameters.

To find an unique solution for this problem another third equations must be specified through equation (3).

$$P = \beta \frac{\sqrt{A} - \sqrt{A_0}}{A_0} \qquad (3)$$

This equation gives a connection between the pressure and the area of a vessel. Additionally the parameters A_0, $\beta = E_0 h_0 \sqrt{(\pi)}$ are set to be constant. A_0 Is the initial value for the area, E_0 is the Young modulus and h_0 is the wall thickness of the arterial segments. The system of differential equations can also be written in a matrix conservative form:

$$\frac{dU}{dt} + \frac{dF}{dz}(U) = B(U), U = [A, Q] F(U) =$$
$$= \begin{pmatrix} Q \\ \alpha \frac{Q^2}{A} + \frac{\beta}{3\rho A_0} A^{3/2} \end{pmatrix} B(U) = \begin{pmatrix} 0 \\ -K_R \frac{Q}{A} \end{pmatrix} \qquad (4)$$

By setting $\alpha = 1$, a flat velocity profile is supposed, which is reasonable for blood stream simulation [1,4,5].

1.2 Boundary conditions

Boundary conditions have to be set to get an unique solution. Here one distinguish between the proximal, namely the inflow and the distal, namely the outflow boundary conditions.

Figure 1 shows a pressure curve which which will be used as the input function for the proximal boundary condition. The pressure of the left ventricle in the heart is approximated by these two sine functions.

For the outflow a Windkessel model consisting of three elements is used. In this case the flow and the area must fulfil the following relation, given trough a ordinary differential equation.
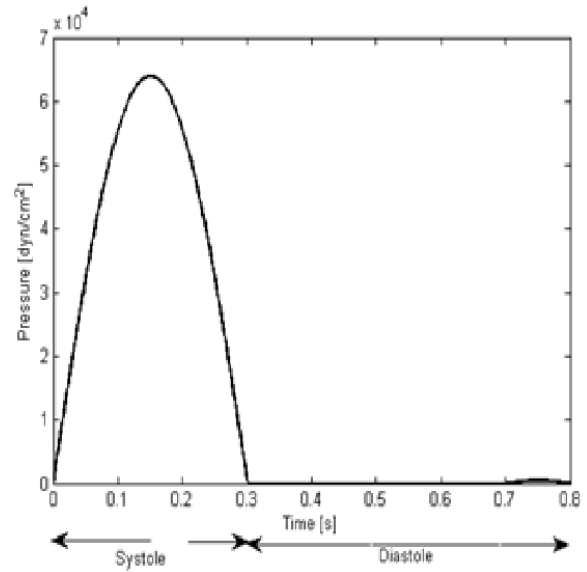

Figure 1: Approximated Pressure Input function for the model.

$$Q(1 + \frac{R_0}{R_1}) + CR_0 \frac{dQ}{dt} = \frac{P}{R_1} + C \frac{dP}{dt} \qquad (5)$$

The Windkessel parameters are the resistances R_0 and R_1 the and the compliance C of the arterial segment [3].

1.3 Bifurcations

A bifurcation is the branching of an artery in two. There are two conditions which has to be fulfilled. Suppose one has an arterial segment (A) which is split up in two arterial segments namely (B) and (C). First of all the condition

$$P^{(A)} = P^{(B)} = P^{(C)} \qquad (6)$$

must be fulfilled, meaning that the pressure is is the same at all three boundaries and secondly that

$$Q^{(A)} = Q^{(B)} + Q^{(C)} \qquad (7)$$

meaning that the flow of the first arterial segment is spilt up in the the flow of the following arterial segments [2].

1.4   The arterial network

The atrial tree consists of 13 central arterial segments.

The rest of the arterial segments will be modelled by choosing the Windkessel parameters in a correct way. Figure 2 shows the abstract arterial tree. The values which will be used are shown later in Table 1 and Table 2 and are taken from physiological data.
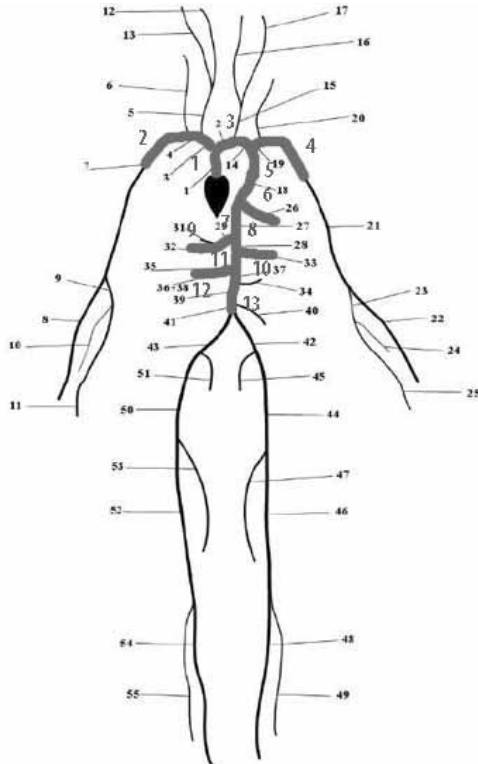

Figure 2: Abstract arterial tree, consisting of the 13 central arteries.

1.5   Taylor-Galerkin method

The system of partial differential equations cannot be solved analytically, that is why the numerical Finite Element Method is used. The basic idea of this method is to do a discretization in time and space.

Lets start with the discretization in time, to do so one uses a Talyor approximation of second order. After this, the discretization in space has to be done. For that the domain $\Omega = [0, L]$, which is namely the arterial segment, is decomposed into N equally spaced elements $[z_i, z_{i+1}]$ and the solution U is discretisized by basic functions of first order through equation (9).

$$U_h = \Sigma_{i=1}^{N} U_i \phi_i \qquad (8)$$

To be more precise for $\phi_i(z)$ first order basic functions or hat functions will be used. Now the weak formulation of the problem has to be calculated. To do so, the term, gained from the second order Taylor approximation, has to be multiplied by a test function and integrated over the domain. After inserting the discretisized solution and discretisized test function the result has the following form:

$$(U_h^{n+1}, \phi_h) = (U_h^n, \phi_h) + \Delta t \left(F^n(U_h^n) + \right.$$
$$\frac{\Delta t}{2} H^n(U_h^n) B^n(U_h^n), \frac{d\phi_h}{dz} \right) \frac{-\Delta t^2}{2}$$
$$(B_U^n(U_h^n) \frac{dF^n}{dz}(U_h^n), \phi_h) \frac{-\Delta t^2}{2} (H^n(U_h^n) \frac{dF^n}{dz}(U_h^n), \frac{d\phi_h}{dz}) \qquad (9)$$
$$+\Delta t (B^n(U_h^n) + \frac{\Delta t}{2} B_U^n(U_h^n) B^n(U_h^n), \phi_h)$$

Equation (9) is the final result and in the implementation the integral terms of the left and the right side will be calculated. This will be done by a numerical quadrature. The implementation is done in MATLAB although some parts of the code are written in C to get more efficient results [7].

1.6   Discretization of boundary conditions

To get unique results boundary conditions must be defined. Before one can do so, the characteristic values have to be calculated. For this system of differential equations one gets the characteristic variables

$$W_{1,2} = \frac{Q}{A} \pm 4(\sqrt{\frac{\beta}{2\rho A_0}} A^{1/4} - c_0). \qquad (10)$$

Because the system of differential equations is a hyperbolic one, only one boundary condition on each side of the arty has to be set. [2]

Proximal boundary conditions

An pressure input function should be used as an input. To do so one manipulate the definition of the inflow characteristic variable to

$$W_1 = W_2^0 + 8\sqrt{\frac{1}{2\rho} p(t) + \frac{\beta}{2\rho\sqrt{A_0}}} - c_0, \qquad (11)$$

so that the pressure $p(t)$ can be used as input directly [7].

Distal boundary conditions

For the distal boundary conditions the Windkessel model is used. The new value of A is found by solving the non-linear equation (12)

$$R_0 \left[\frac{Q_R}{A_R} + 4c_1(A_R) \right] A - 4R_0 c_1(A)A -$$
$$-P(A) + P_C^n = 0, \tag{12}$$

where P_C is calculated through (14)

$$P_C^n = P_C^{n-1} + \frac{\Delta t}{C}(Q_R - \frac{P_C^{n-1}}{R_1})P_C^{n-1}$$
$$= P(A_R) - Q_R R_0. \tag{13}$$

To solve this non-linear equation a Newton method with start value $A = A_R$ is used. A_R and Q_R are the values at the end of the arterial segment. The new value of Q is then found through equation (14). [7]

$$Q = \frac{P(A) - P_C}{R_0} \tag{14}$$

1.7 Boundary conditions at the bifurcations

When one artery is split up into two others, the two conditions from equations (6) and (7) has to be fulfilled. This is done by solving a system of six non-linear equations given through (15). [7]

$$Q^{(A)} = Q^{(B)} + Q^{(C)}$$

$$W_1^{(A)} = \frac{Q^{(A)}}{A^{(A)}} + 4\left(\sqrt{\frac{\beta^{(A)}}{2\rho A_0^{(A)}}} (A^{(A)})^{1/4} - c_0^{(A)} \right)$$

$$W_2^{(B)} = \frac{Q^{(B)}}{A^{(B)}} - 4\left(\sqrt{\frac{\beta^{(B)}}{2\rho A_0^{(B)}}} (A^{(B)})^{1/4} - c_0^{(B)} \right)$$

$$W_2^{(C)} = \frac{Q^{(C)}}{A^{(C)}} - 4\left(\sqrt{\frac{\beta^{(C)}}{2\rho A_0^{(C)}}} (A^{(C)})^{1/4} - c_0^{(C)} \right) \tag{15}$$

$$\beta^{(A)} \frac{\sqrt{A^{(A)}} - \sqrt{A_0^{(A)}}}{A_0^{(A)}}$$

$$= \beta^{(B)} \frac{\sqrt{A^{(B)}} - \sqrt{A_0^{(B)}}}{A_0^{(B)}} \beta^{(A)} \frac{\sqrt{A^{(A)}} - \sqrt{A_0^{(A)}}}{A_0^{(A)}}$$

$$= \beta^{(C)} \frac{\sqrt{A^{(C)}} - \sqrt{A_0^{(C)}}}{A_0^{(C)}}$$

2 Implementation

In this section, a part of the code is presented and described. This part realizes the bifurcations in the arterial tree. As a starting point one has a matrix (Tree) consisting of the numbers of the parent and the child arteries.

First of all, out of this matrix the connection are read out.

```
Place_1=Tree(:,1);
Place_2=Tree(:,2);
Place_3=Tree(:,3);

bif=@(ub)[
ub(2)/ub(1)+4*(sqrt(beta(Place_1)
/(2*rho*A0(Place_1)))*(ub(1))^(1/4)-
c0(Place_1))-W1L(1,1,Place_1);...

ub(4)/ub(3)-
4*(sqrt(beta(Place_2)/(2*rho*A0(Place_2)))*(u
b(3))^(1/4)-c0(Place_2))-W20(1,1,Place_2);...

ub(6)/ub(5)-
4*(sqrt(beta(Place_3)/(2*rho*A0(Place_3)))*(u
b(5))^(1/4)-c0(Place_3))-W20(1,1,Place_3);...

ub(4)+ub(6)-ub(2);...

beta(Place_2)*(sqrt(ub(3))-
sqrt(A0(Place_2)))/A0(Place_2)-
beta(Place_1)*(sqrt(ub(1))-
sqrt(A0(Place_1)))/A0(Place_1);...

beta(Place_3)*(sqrt(ub(5))-
sqrt(A0(Stelle_3)))/A0(Place_3)-
beta(Place_1)*(sqrt(ub(1))-
sqrt(A0(Place_1)))/A0(Place_1)];
```

After setting up the six nonlinear equations (bif) taken from equation (16), the system can be solved with the MATLAB function fsolve and the new boundary values can be set.

```
[Ub(:)]=fsolve(bif,Ub);

uAL(Place_1)=(Ub(1));
uQL(Place_1)=(Ub(2));
uA1(Place_2)=(Ub(3));
uQ1(Place_2)=(Ub(4));
uA1(Place_3)=(Ub(5));
uQ1(Place_3)=(Ub(6));
```

3 Results

The parameters which where used for the simulation are shown in Table 1 and Table 2. The unit for the radius and length are cm, for the resistances is $\frac{dyns}{cm^5}$, for the compliance it is $\frac{cm}{dyn} 10^{-5}$ and for the Young modulus is $\frac{dyns}{cm^2} 10^6$.

Parameters	Values	Unit
ρ	1	g/cm^3
h_0	0.15	cm
β	$E_0 * h_0 * \sqrt{(\pi)}$	dyn/cm
K_R	$8\pi\nu$	$Poise$
ν	0.035	$Poise$
dt	$2 * 10^{-5}$	sec
$time$	0.8	sec

Table 1: Values of the parameters used for the simulation .

Artery	Child	r	l	R0	R1	C	E0
1	2 \ 3	1.4	4	-	-	-	4
2	-\ -	0.6	3	50	1000	8	1.8
3	4 \ 5	1.3	2	-	-	-	3.8
4	-\ -	0.4	6	60	1200	8	1.2
5	6 \ 7	1	4	-	-	-	3.5
6	-\ -	0.3	5	70	1400	9	0.9
7	8 \ 9	0.9	3	-	-	-	3.2
8	10 \ 11	0.8	5	-	-	-	2.9
9	-\ -	0.3	6	90	1600	9	0.9
10	-\ -	0.2	7	90	1600	9	0.7
11	12 \ 13	0.6	2	-	-	-	2.7
12	-\ -	0.4	6	100	1800	8.2	1.2
13	-\ -	0.5	2	100	1800	8.2	2.4

Table 2: Artery parameter used for the simulation of the arterial tree .

For every arterial segment one gets the pressure, the flow, the velocity and the cross-section area as an output.
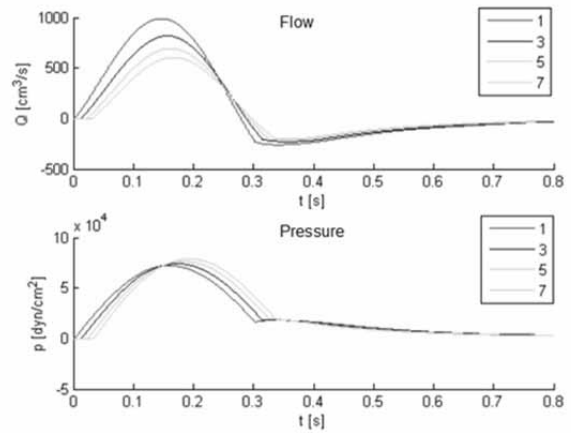

Figure 4: Flow and pressure results for the first, the third, the fifth and the seventh arterial segment. The values are read out in the middle of the arterial segments.

First of all the results from a way through the arterial tree, from Figure 2, is presented. The way goes from the first to the third, then to the fifth and finally in the seventh arterial segment. Figure 4 shows the results for the pressure and the flow in these arterial segments taken at the midpoint of the arterial segments.

Two effects can be seen: First that the pressure level rises and secondly the flow is decreasing from arterial segment to arterial segment.
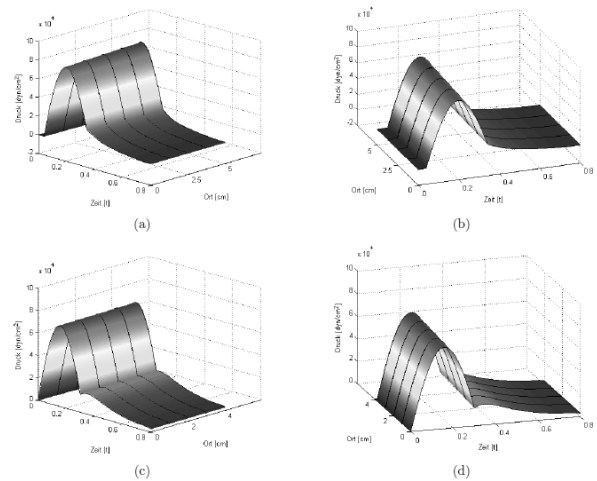

Figure 5: Pressure-position-time plot of the eight arterial segment (a)-(b) and the first arterial segment (c)-(d) in two different perspectives.

Figure 5 shows the pressure-position-time plot of the eight arterial segment from two different perspectives (a) and (b) and the the pressure-position-time plot of the first arterial segment from two different perspectives (d) and (c). Here one can observe again the influence of the Windkessel model and also the time dependent delay in place can be seen in Figure 5 (b) and (d).

4 Conclusion and Outlook

In comparison to already accomplished models and physiologically data the gained results with this model are very similar. That is why it can be concluded that the application of a one dimensional Finite Element Method approach along with the particular implementation presented can describe the effects in a system of human arteries in a realistic way.

A field of application for this model is the early diagnosis of cardiovascular diseases. With measurements gained from healthy patients, the model can be parameterized. The calculations from this model can be compared with measurements from patients with cardiovascular diseases in order to conclude about abnormal changes in the cardiovascular system [6].

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