Short Remark on Lateral Vibration of Functionally Graded Beams

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Abstract. In his study Aydogdu analyzes the vibration of axially functionally graded simply supported beams. The main idea behind his calculation is that the vertical displacement is supposed to have a special form, which transforms the Euler-Bernoulli differential equation for the motion for the lateral vibrations into an exact linear differential equation which depends on the axial coordinate only. In this paper we generalize the method used by Aydogdu and determine the largest function class of the form $w(x,t) = F(x) \cdot G(t)$ for which the same method is applicable.

Introduction

Finding closed-form solutions for the vibration and buckling of the beams has been in the focus of scientific engineering research for a long time. In general case, to reach this goal appears to be not realistic. Several special cases have been examined. A comprehensive summary of the subject can be found in [2]. Further examples can be found in [3], and [4]. The detailed description of usage and programming of Maple can be found in [5] and [6].

1 Basic Model

This article refers to the study of Aydogdu ([1]) in which the equation of motion for the lateral vibrations of axially functionally graded simply supported beams is examined by using the semi-inverse method. The Euler-Bernoulli differential equation for the lateral vibrations of FG beams has the form:

$$\frac{\partial^2}{\partial x^2}\left(E(x)J\left(\frac{\partial^2}{\partial x^2}w(x,t)\right)+\rho A\left(\frac{\partial^2}{\partial t^2}w(x,t)\right)\right) \quad (1)$$

where ρ is the density, A is the cross sectional area, w is the transverse deflection, J is the moment of inertia and $E(x)$ is the elasticity modulus of the beam and t is the time. The cross section area A and the moment of inertia J are assumed to be constant.

In [1] Aydogdu supposes that the vertical displacement has a special form $w(x,t) = W_m\, sin(\beta x)\, sin(\omega t)$ and points out that the substitution of this form into the Euler-Bernoulli differential equation above transforms it into an exact linear differential equation which depends on the axial coordinate only.

The reconstruction of this calculation is straightforward. Consider the Euler-Bernoulli differential equation of motion for the lateral vibrations and perform the partial derivations.

$$> \frac{\partial^2}{\partial x^2}\left(E(x)\,J\left(\frac{\partial^2}{\partial x^2}w(x,t)\right)+\rho\,A\left(\frac{\partial^2}{\partial t^2}w(x,t)\right)\right)=0$$

$> value(\%)$

$$\frac{d^2}{dx^2}(E(x))\,J\left(\frac{\partial^2}{\partial x^2}w(x,t)\right)+2\left(\frac{d}{dx}E(x)\right)J\left(\frac{\partial^3}{\partial x^3}w(x,t)\right)+ +E(x)\,J\left(\frac{\partial^4}{\partial x^4}w(x,t)\right)+\rho\,A\left(\frac{\partial^2}{\partial t^2}w(x,t)\right)=0 \quad (2)$$

Next suppose that the vertical displacement is the product of two sinus functions, more specifically let

$$> w(x,t) = W_m\, sin\left(\frac{m\pi x}{L}\right) sin(\omega t) \quad (3)$$

where W_m is the amplitude of the vibrations, m is the half wave number, L is the length and ω is the radial natural frequency of the FG beam. Introducing the notation $\beta = \frac{m\pi}{L}$ we obtain:

$> algsubs(\frac{m\pi}{L} = \beta, \%)$

$$w(x,t) = W_m\, sin(\beta x)\, sin(\omega t) \tag{4}$$

Substituting (4) into (2) the resulting differential equation does not depend on the variable t.

$> eval((2),(4)$

$$\begin{aligned} &-\left(\frac{d^2}{dx^2}E(x)\right) J\, W_m\, sin(\beta x)\, sin(\omega t) - \\ &-2\left(\frac{d}{dx}E(x)\right) J\, W_m\, cos(\beta x)\, \beta^3\, sin(\omega t) + \\ &+E(x)\, J\, W_m\, sin(\beta x)\, \beta^4\, sin(\omega t) - \rho\, A sin(\beta x)\, sin(\omega t)\, \omega^2 = 0 \end{aligned} \tag{5}$$

$> expand\left(\frac{(5)}{J\, W_m\, sin(\omega t)\, \beta^2}\right)$

$$\begin{aligned} &-\left(\frac{d^2}{dx^2}E(x)\right) sin(\beta x) - 2\beta\left(\frac{d}{dx}E(x)\right) cos(\beta x) + \\ &+\beta^2 E(x)\, sin(\beta x) - \frac{\rho\, A sin(\beta x)\, \omega^2}{J\, \beta^2} = 0 \end{aligned} \tag{6}$$

Note that the function $w(x,t)$ in (3) has the form $w(x,t) = F(x)\cdot G(t)$. This observation naturally raises the question: how should we choose the functions $F(x)$ and $G(t)$ so that the choice $w(x,t) = F(x)\cdot G(t)$ results in a differential equation which does not depend on variable t? In other words, denote the class of all functions of the form $F(x)\cdot G(t)$ by C and determine the largest subset of C whose elements transform the Euler-Bernoulli differential equation into a DE which depends on variable x only.

2 Generalization

Proposition 1. For arbitrary function $F(x)$ and for the function

$$G(t) = A\, sin(\omega t) + B\, cos(\omega t)$$

the choice

$$w(x,t) = F(x)\,(A\, sin(\omega t) + B\, cos(\omega t))$$

transforms the Euler-Bernoulli differential equation into a DE, which does not depends on variable t.

Proof. Suppose that $w(x,t)$ has the desired form and substitute it into the Euler-Bernoulli differential equation.

$$> w(x,t) = F(x)\ (A\, sin(\omega t) + B\, cos(\omega t)): \tag{7}$$

$> eval((1),(7))$

$$\begin{aligned} &\left(\frac{d^2}{dx^2}E(x)\right) J\left(\frac{d^2}{dx^2}F(x)\right)(A\sin(\omega t) + B\cos(\omega t)) + \\ &+2\left(\frac{d}{dx}E(x)\right) J\left(\frac{d^3}{dx^3}F(x)\right)(A\sin(\omega t) + B\cos(\omega t)) + \\ &+E(x)\, J\left(\frac{d^4}{dx^4}F(x)\right)(A\sin(\omega t) + B\cos(\omega t)) - \\ &-\rho\, A\, F(x)\, \omega^2\, (A\sin(\omega t) + B\cos(\omega t)) = 0 \end{aligned} \tag{8}$$

Freeze the subexpression $(A\sin(\omega t) + B\sin(\omega t))$ and divide the resulting equation by $J\alpha$, provided $\alpha \neq 0$.

$> algubs(A\sin(\omega t) + B\cos(\omega t) = \alpha, (8))$

$$\begin{aligned} &\left(\frac{d^2}{dx^2}E(x)\right) J\left(\frac{d^2}{dx^2}F(x)\right)\alpha + 2\left(\frac{d}{dx}E(x)\right) J\left(\frac{d^3}{dx^3}F(x)\right)\alpha + \\ &+E(x)\, J\left(\frac{d^4}{dx^4}F(x)\right)\alpha - \rho\, A\, F(x)\, \omega^2\alpha = 0 \end{aligned} \tag{9}$$

$> expand(\frac{\%}{J\alpha})$

$$\begin{aligned} &\left(\frac{d^2}{dx^2}E(x)\right)\left(\frac{d^2}{dx^2}F(x)\right) + 2\left(\frac{d}{dx}E(x)\right)\left(\frac{d^3}{dx^3}F(x)\right) + \\ &+E(x)\left(\frac{d^4}{dx^4}F(x)\right) - \frac{\rho\, A\, F(x)\, \omega^2}{J} = 0 \end{aligned} \tag{10}$$

This proofs Proposition 1.

Although this is not in the focus of our investigations the next proposition determines the general solution of DE.

Propostion 2. The general solution of DE (10) is

$$E(x) = \frac{\_C2 + \_C1\, x + \frac{\rho\, A\, F(x)\, \omega^2}{J}\int\int F(x)dx\, dx}{\frac{d^2}{dx^2}F(x)}$$

Proof. The proof is a simple three step calculation. Maple is used to evaluate the differential equation above after the substitution the supposed value of the function $E(x)$. The resulting expression is huge and far from being perspicuous. This does not mean, however, that Maple cannot simplify it to zero.

$$E(x) = \frac{\_C2 + \_C1\, x + \frac{\rho\, A\, F(x)\, \omega^2}{J}\int\int F(x)dx\, dx}{\frac{d^2}{dx^2}F(x)}: \tag{11}$$

$> eval((10),\%)$

$$\left(\frac{\rho\,A\,F(x)\omega^2}{\left(\frac{d^2}{dx^2}F(x)\right)J}-\frac{2\left(\_C1\,J+\rho\,A\,\omega^2\left(\int F(x)dx\right)\right)\left(\frac{d^3}{dx^3}F(x)\right)}{\left(\frac{d^2}{dx^2}F(x)\right)^2 J}+\right.$$
$$+\frac{2\left((\_C2+\_C1\,x)J+\rho\,A\,\omega^2\left(\int\int F(x)dx\,dx\right)\right)\left(\frac{d^3}{dx^3}F(x)\right)^2}{\left(\frac{d^2}{dx^2}F(x)\right)^3 J}-$$
$$\left.-\frac{\left((\_C2+\_C1\,x)J+\rho\,A\,\omega^2\left(\int\int F(x)dx\,dx\right)\right)\left(\frac{d^4}{dx^4}F(x)\right)}{\left(\frac{d^2}{dx^2}F(x)\right)^2 J}\right)\cdot$$
$$\cdot\left(\frac{d^2}{dx^2}F(x)\right)+2\left(\frac{\_C1\,J+\rho\,A\,\omega^2\left(\int F(x)dx\right)}{\left(\frac{d^2}{dx^2}F(x)\right)J}-\right. \tag{12}$$
$$\left.-\frac{\left((\_C2+\_C1\,x)J+\rho\,A\,\omega^2\left(\int\int F(x)dx\,dx\right)\right)\left(\frac{d^3}{dx^3}F(x)\right)}{\left(\frac{d^2}{dx^2}F(x)\right)^2 J}\right)\cdot$$
$$\cdot\left(\frac{d^3}{dx^3}F(x)\right)+$$
$$+\frac{\left((\_C2+\_C1\,x)J+\rho\,A\,\omega^2\left(\int\int F(x)dx\,dx\right)\right)\left(\frac{d^4}{dx^4}F(x)\right)}{\left(\frac{d^2}{dx^2}F(x)\right)J}\cdot$$
$$\cdot\left(\frac{d^2}{dx^2}F(x)\right)-\frac{\rho\,A\,F(x)\omega^2}{J}=0$$

$> simplify((\%)$

$$0=0 \tag{13}$$

In the end we show the reverse of Proposition 1. In other words, we prove that the form $G(t)=A\sin(\omega t)+B\cos(\omega t)$ is not only sufficient but also necessary condition for fact that the choice $w(x,t)=F(x)\cdot G(t)$ transforms the Euler-Bernoulli differential equation into a DE, which does not depend in variable t.

Propostion 3. If

$$w(x,t)=F(x)\cdot G(x)$$

and its substitution transforms the Euler-Bernoulli differential equation into a DE, which does not depend in variable t, then $G(t)$ must have the form

$$G(t)=A\sin(\omega t)+B\cos(\omega t).$$

Proof. Consider the function

$> w(x,t)=F(x)\,G(t)$

$$w(x,t)=F(x)\,G(t) \tag{14}$$

and let us substitute it into

$> eval((1),(14))$

$$\left(\frac{d^2}{dx^2}E(x)\right)J\left(\frac{d^2}{dx^2}F(x)\right)G(t)+2\left(\frac{d}{dx}E(x)\right)J\left(\frac{d^3}{dx^3}F(x)\right)\cdot$$
$$\cdot G(t)+E(x)\,J\left(\frac{d^4}{dx^4}F(x)\right)G(t)+\rho\,A\,F(x)\left(\frac{d^2}{dt^2}G(t)\right)=0 \tag{15}$$

All terms on the left hand side of this equation is divisible by $G(t)$ except for the last one. Divide the equation by $J\,G(t)$ provided that $G(t)\neq 0$.

$> expand(\frac{\%}{J\,G(t)})$

$$\left(\frac{d^2}{dx^2}E(x)\right)\left(\frac{d^2}{dx^2}F(x)\right)+2\left(\frac{d}{dx}E(x)\right)\left(\frac{d^3}{dx^3}F(x)\right)+$$
$$+E(x)\left(\frac{d^4}{dx^4}F(x)\right)+\frac{\rho\,A\,F(x)\left(\frac{d^2}{dt^2}G(t)\right)}{J\,G(t)}=0 \tag{16}$$

The first three terms and the coefficient of the second derivative of $G(t)$ in the numerator of the fourth term do not depend on variable t, which yields that the equation above can be written in the form

$$> A+\frac{B\left(\frac{d^2}{dx^2}G(t)\right)}{G(t)}=0: \tag{17}$$

The solution of this differential equation can be easily determined by means of procedure *dsolve*.

$> dsolve(\%,G(t))$

$$G(t)=\_C1\sin\left(\frac{\sqrt{A}}{\sqrt{B}}t\right)+\_C2\cos\left(\frac{\sqrt{A}}{\sqrt{B}}t\right) \tag{18}$$

Introducing the notation $\omega^2=\frac{A}{B}$ we obtain the desired form.

$> G(t)=algsubs(\frac{\sqrt{A}}{\sqrt{B}}t=\omega\,t,rhs(\%))$

$$G(t)=A\sin(\omega t)+B\cos(\omega t)$$

This proofs Proposition 3.

3 Conclusion

The aim of this paper has been to show the usage of Maple general purpose computer algebraic system in the scientific engineering calculations. We have entrusted the performance of all calculation step to Maple. In this way we have used it not only to convert different complex expressions but Maple turned out to be a useful tool in the proofs of propositions.

We have pointed out that the largest class of functions of the form $w(x,t) = F(x) \cdot G(t)$, which transforms the Euler-Bernoulli differential equation for the lateral vibrations of FG beams into an exact linear differential equation depending on the axial coordinate only, consists of the functions $G(t) = F(x)(A\sin(\omega t) + B\cos(\omega t))$. We have also determined closed form solution of the transformed DE.

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