Mathematical Framework for Graphical Simulation Models of Dynamical Systems in SIMULINK

Andreas Körner

Institute for Analysis and Scientific Computing, Vienna University of Technology, Wiedner Haupstraße 8-10, 1040 Vienna, Austria; *andreas.koerner@tuwien.ac.at*

Simulation Notes Europe SNE 23(3-4), 2013, 123 - 128
DOI: 10.11128/sne.23.tn.10203
Received: September 15, 2013; Revised: November 15, 2013;
Accepted: November 30, 2013;

Abstract. This article is about the mathematical description, analysis and background of the simulation environment SIMULINK. This simulation environment is a well-known tool in applied mathematics and a wide range of fields in engineering, mainly control engineering. SIMULINK is driven by control engineering which is recognisable in the block oriented structure as well as in the notation and characterisation of simulation models. This article will give an overview about the mathematics behind the simulation models in SIMULINK and discuss several items. The structure of the article starts in the beginning with the mathematical definitions and context. A relation between the mathematical aspects and the situation in the simulation environment is given.

Introduction

As an introduction of the article the setup of dynamical systems will be given and discussed. Mainly dynamical systems are linked for many people with ordinary differential equations, initial vale problems or partial differential equation in more complicated modelling approaches. This link is wrong in general, a dynamical system can be defined without a differential equation. This allows to consider a wider range of dynamical systems, e.g. time continues and time discrete. In the case of time continuous systems the connection to differential equations can be established a shown in the following steps.

Dynamical System. Assume sets T and $X \neq \emptyset$ with $T \in \{\mathbb{N}_0, \mathbb{Z}, \mathbb{R}_0^+, \mathbb{R}\}$. Furthermore define a mapping $g\colon T \times X \to X$ which satisfies the for $x \in X$ and $t_1, t_2 \in T$

1. $g(0,t) = x$,
2. $g(t_1, g(t_2, x)) = g(t_1 + t_2, x)$.

The triple (T, X, g) is named a dynamical system, in case of $T \in \{\mathbb{N}_0, \mathbb{Z}\}$ it is called time discrete, in case of $T \in \{\mathbb{R}_0^+, \mathbb{R}\}$ it is called time continuous. X is called the state space and g the flow. For easier notation g is redefined to

$$g_t(x) = g(t,x). \tag{1}$$

It is easy to proof that for $t, t_1, t_2 \in T$ the mapping g fulfills

1. $g_0 = \mathrm{id}$,
2. $g_{t_1+t_2} = g_{t_1} \circ g_{t_2} = g_{t_2} \circ g_{t_1}$,
3. $g_t^{-1} = g_{-t}$.

Time Continuous Dynamical Systems. Assume $x\colon \mathbb{R} \to \mathbb{R}^n$, $x \in \mathscr{C}^1(\mathbb{R})$ and g continues differentiable. With $x(0) = x_0$ and $x(t) = g_t(x_0)$ let's consider

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}x(t) &= \lim_{h\to 0}\frac{1}{h}\big(g_{t+h}(x_0) - g_t(x_0)\big) = \\
&= \left(\left(\lim_{h\to 0}\frac{1}{h}(g_h - \mathrm{id})\right) \circ g_t\right)(x_0) = \\
&= \left(\frac{\partial}{\partial t} g_t\Big|_{t=0} \circ g_t\right)(x_0) = \\
&= \left(\frac{\partial}{\partial t} g_t\Big|_{t=0}\right)\big(x(t)\big).
\end{aligned} \tag{2}$$

Time continuous dynamical systems with conditions regarding x and g assumed above can be described by

$$x'(t) = f\big(x(t)\big),\ f\big(x(t)\big) = \left(\frac{\partial}{\partial t} g_t\Big|_{t=0}\right)\big(x(t)\big). \tag{3}$$

1 Model Structure in SIMULINK

Let's consider an arbitrary SIMULINK model, e.g. as illustrated in Figure 1.

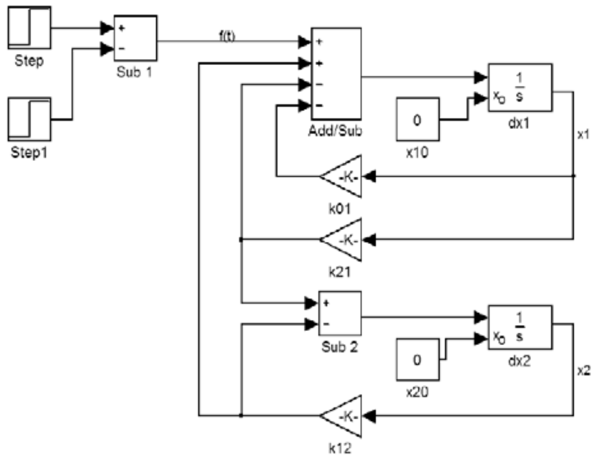

Figure 1: An arbitrary SIMULINK model to illustrate different relations between certain SIMULINK elements.

The model structure shows two issues:

1. input-output relation of each component
2. graph structure defines the the topology of the component connection

The following sections will discuss this two aspects, the signal flow graphs as a mathematical environment to describe the interconnection and the input-output relation of the individual blocks to implement different behavior.

2 Graph Concept for SIMULINK Models

Oriented Graphs and SIMULINK-Models. Consider $V = V(G)$ as the set of nodes and $E = E(G)$ the set edges. The tuple $G = (V, E)$ is called a graph and $e \in E(G)$ an edge. The edge is called oriented if e is represented as an ordered pair $e = \langle v_1, v_2 \rangle$ of nodes $v_1, v_2 \in V(G)$, the starting node v_1 and the end node v_2. The edge is called not oriented if the edge is represented via a not oriented pair $e = (v_1, v_2)$ of nodes $v_1, v_2 \in V(G)$.

As illustrated in Figure 1 each SIMULINK-model has an orientation. For this purpose the oriented graph is a suitable mathematical environment to represent the orientation in a SIMULINK-model. Next step is to express the mathematical and technical manipulations in the model, which are represented in a block with certain input and output signals or vectors of signals.

Weighted Graphs and Signal Flow Graphs. Let'S assume $G = (V, E)$ to be a graph and $w\colon E \to \mathbb{R}_0^+$. The triple (V, E, w) is called weighted graph and w the weighting function. Let (V, E, w) be a weighted graph and (e_k) a series of edges of the graph. The weight of the series of edges is defined as

$$w\big((e_k)\big) = \sum_{i=1}^{k} w(e_k). \tag{4}$$

Furthermore the distance between two nodes in the graph is defined as the minimum of weights thru all series of nodes which are connecting the two discussed nodes. The illustration of a particle of a weighted graphs is illustrated in Figure 2.

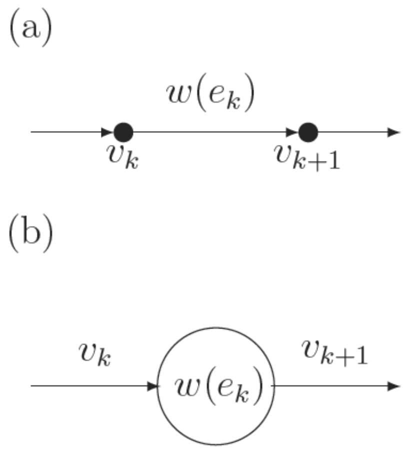

Figure 2: An illustration of a particle of a weighted graph, (a) classical representation in graph theory, (b) representation focusing on signal flow graphs.

The last step in the mathematical environment is the integration of signal flow in the graph concept.

Assume $G_w = (V, E, w)$ as an oriented and weighted graph wit the set of nodes $V = \{v_1, \ldots, v_k\}$, the set of edges $E = \{e_1, \ldots, e_m\}$ and the weighting function $w\colon E \to \mathbb{R}_0^+$. Furthermore is $X = \{x_1, \ldots, x_k\}$ with $x_i \in \mathbb{R}$ for $i = 1, \ldots, k$ defined. The mapping

$$\mu\colon V \to X,\ v_i \mapsto x_i$$

relates each node v_i with a value x_i. The values $x_j = \mu(v_j)$, $x_\ell = \mu(v_\ell)$ of X are connected to w by the equation

$$x_\ell = x_j \cdot w(e_j), \tag{5}$$

if $e_j = \langle v_j, v_\ell \rangle$ is valid. If the node v_ℓ is the end of more than one edge e_j, e.g. $r \in \mathbb{N}$

$$e_j = \langle v_j, v_\ell \rangle,\ e_{j+1} = \langle v_{j+1}, v_\ell \rangle,\ \ldots, e_{j+r} = \langle v_{j+r}, v_\ell \rangle,$$

than for x_ℓ hold

$$x_\ell = \sum_{s=0}^{r} x_{j+s} \cdot w(e_{j+s}).$$

The tuple $S = (G_w, \mu)$ is called a signal flow graph.

To have in mind that this mathematical environment is designed for a description for SIMULAINK-models this definition of a signal flow graph is not generalized enough to represent more than linear models in SIMULINK. For this purpose the concept has to be redesigned to cover more general model structures.

Generalised Signal Flow Graphs. In the definition given up to now the relation between two nodes x_j and x_ℓ is given by the weighting function w along the edge e_j according to

$$x_j = x_\ell \cdot w(e_j).$$

This limitation in the mathematical description will not allow to include all SIMULINK-models, so the definition need a more general approach. The fist step is to generalize the weighting.

Assume S as a signal flow graph. The mapping

$$w_{e_j} \colon X \to X,\ x_j \mapsto x_\ell = w_{e_j}(x_j) \tag{6}$$

defines the so-called generalised weighting function of the signal flow graph S.

This definition of a generalised weighting function allows to cover a wider range of relations between nodes in the signal flow graph. If there are no possibilities of misunderstanding the shorter notation

$$x_\ell = w(x_j)$$

can be applied.

The next step of the generalisation in the signal flow graph concept addresses the subject of signals. The definition up to now didn't cover the flow of signals, only the relation was oriented which implied a flow. This will be improved in the following extension.

Assume a set $\Omega \subseteq \mathbb{R}$ and a function $f \colon \Omega \to \mathbb{R}$.

1. The function f is called a signal, if the characteristics over the time $t \in \Omega$ covers some information of a physical quantity.
2. A signal is called causal, if $f(t) = 0$ for all $t < 0$t. More general also $f(t) = 0$ for all $t < t_0$ possible.
3. The set
$$\mathrm{L}^2(\Omega) = \left\{ f \colon \Omega \to \mathbb{R} \colon \int_\Omega |f(t)|^2 \,\mathrm{d}t < \infty \right\}$$
is called the set of quadratic integrable functions.
4. $(\mathrm{L}^2(\Omega), +, \cdot)$ with the composition
is a vector space. With
$$\langle f, g \rangle_{\mathrm{L}^2} = \int_\Omega f(t) g(t) \,\mathrm{d}t$$
a scalar product is defined and its induced norm
$$\|f\|_{\mathrm{L}^2} = \sqrt{\langle f, f \rangle_{\mathrm{L}^2}} = \sqrt{\int_\Omega |f(t)|^2 \,\mathrm{d}t}.$$
For that reason the vector space $(\mathrm{L}^2(\Omega), +, \cdot)$ is regarding $\|\cdot\|_{\mathrm{L}^2}$ complete and so a Hilbert-space.

This mathematical excursion brings the definition to the second most important generalisation related to the signal space. The Hilbert-space $(\mathrm{L}^2(\Omega), +, \cdot)$ with the scalar product $\langle \cdot, \cdot \rangle_{\mathrm{L}^2}$, whose elements are signals, is called signal space. Also for this vector space the notation L^2 is used, for causal signals for example the notation would be $\mathrm{L}^2(\mathbb{R}_0^+)$.

Finally the introduced concepts are combined in the definition of the generalised signal flow graph.

Assume $G_w = (V, E, w)$ as an oriented and weighted graph with the set of nodes $V = \{v_1, \ldots, v_k\}$, the set of edges $E = \{e_1, \ldots, e_m\}$ and the weighting function $w \colon E \to \mathbb{R}_0^+$. Moreover are $x_i \in \mathrm{L}^2(I)$ for $i = 1, \ldots, k$ and $I \subseteq \mathbb{R}$. The mapping

$$\mu \colon V \to X,\ v_i \mapsto x_i$$

relates each node v_i with a corresponding signal x_i. The values $x_j = \mu(v_j)$, $x_\ell = \mu(v_\ell)$ of the set X are associated via w referred to

$$x_\ell(t) = w_{e_j}\big(x_j(t)\big),$$

if $e_j = \langle v_j, v_\ell \rangle$ is valid.
Is v_ℓ the end node of more than one edge e_j, so for $r \in \mathbb{N}$

$$e_j = \langle v_j, v_\ell \rangle,\ e_{j+1} = \langle v_{j+1}, v_\ell \rangle,\ \ldots, e_{j+r} = \langle v_{j+r}, v_\ell \rangle,$$

for x_ℓ the relation

$$x_\ell = \sum_{s=0}^{r} w_{e_{j+s}}(x_{j+s})$$

is valid. The graph $S = (G_w, \mathrm{L}^2(I), \mu)$ is called a generalised signal flow graph.

In simulation environments the mathematical description has a disfigurement. In computer the L^2 formulation is not representable. By sampling of the signals the set X can be constructed and a purely discrete graph is used for the numerical simulation.

The interval $I = [a,b] \subset \mathbb{R}$ with $a < b$ and a signal $x \in \mathrm{L}^2(I)$ will be considered. Moreover $\mathscr{Z} = \{t_1, t_2, \ldots, t_{k-1}, t_k\}$ is a segmentation of I with $t_1 = a$ and $t_k = b$. Is $t_{i+1} - t_i = \Delta t$ for all $i = 1, \ldots, k$ the segmentation is called equidistant. Via the determination

$$x(t_i) = x_i$$

it is defined a mapping $\delta_a\colon \mathrm{L}^2(I) \to \mathbb{R}$, which is called the discretisation of I. This mapping is the connection point between the generalized signal flow graph - the continues mathematical framework - and the regular signal flow graph - the numerical setup for system simulation. The link between is given by the mapping

$$\delta_a\colon \mathrm{L}^2(I) \to \mathbb{R},\ x \mapsto x(t_i)$$

and the construction of the set X by

$$X = \{x_1 = x(t_1), x_2 = x(t_2) \ldots, x_k = x(t_k)\}.$$

3 Input-Output Relation

SIMULINK is designed by definition via input-output relations. In linear cases the input-output scheme is mathematical well observed and there are several theories available for the model analysis. In the nonlinear case the mathematical toolbox is smaller or the available methods and theorems are not that global and general as it is in the linear domain. The following subsections will discuss this difference.

3.1 Linear Time Invariant Systems

Time Continuous Systems. Linear time invariant systems are described via a charming mathematical environment - the Laplace transform. This is an integral transform of the form

$$\mathscr{L}(f)(s) = \int_0^\infty f(t)\mathrm{e}^{-st}\,\mathrm{d}t. \tag{7}$$

Only signals of exponential order, this means that $s_0 > 0$ and $M > 0$ exist that satisfy for $T > 0$ the condition

$$|f(t)| \leq M \cdot \mathrm{e}^{s_0 t} \tag{8}$$

for all $t > T$.

The standard description of LTI–systems in time domain is given by the state space description

$$\begin{aligned} x'(t) &= A\,x(t) + b\,u(t), \qquad x(0) = x_0 \\ y(t) &= c^T x(t) + d\,u(t), \end{aligned} \tag{9}$$

with $b, c, x \in \mathbb{R}^n$, $A \in \mathbb{R}^{n \times n}$ and $u, y, d \in \mathbb{R}$. This state space description is equivalent to a ordinary differential equation n–th order. The Laplace transform lead to a system description in the Laplacian domain.

Assume $U = \mathscr{L}(u)$ the Laplacian of the input signal and $Y = \mathscr{L}(y)$ the Laplacian of the output signal of a LTI-system with the state $x \in \mathbb{R}^n$ and $x_0 = 0$. The function $G\colon \mathbb{C} \to \mathbb{C}$ which fulfills the relation

$$Y(s) = G(s) \cdot U(s) \tag{10}$$

for each input signal is called the transfer function of the LTI-system.

This mathematical environment offers a structure which is called an algebra. Interpreted with the transfer-function, represented as blocks in the signal flow graph, a particular model can be built by using a certain topology of transfer function blocks. Some basic circuit arrangements are illustrated in Figure 3.

Discrete Time Systems. The discretisation introduced in the section above leads also to the description of discrete linear time invariant systems. The equivalent to the Laplace transform in the discrete time is the Z–Transform. This transformation is as well linear and is applied on series' instead of signals.

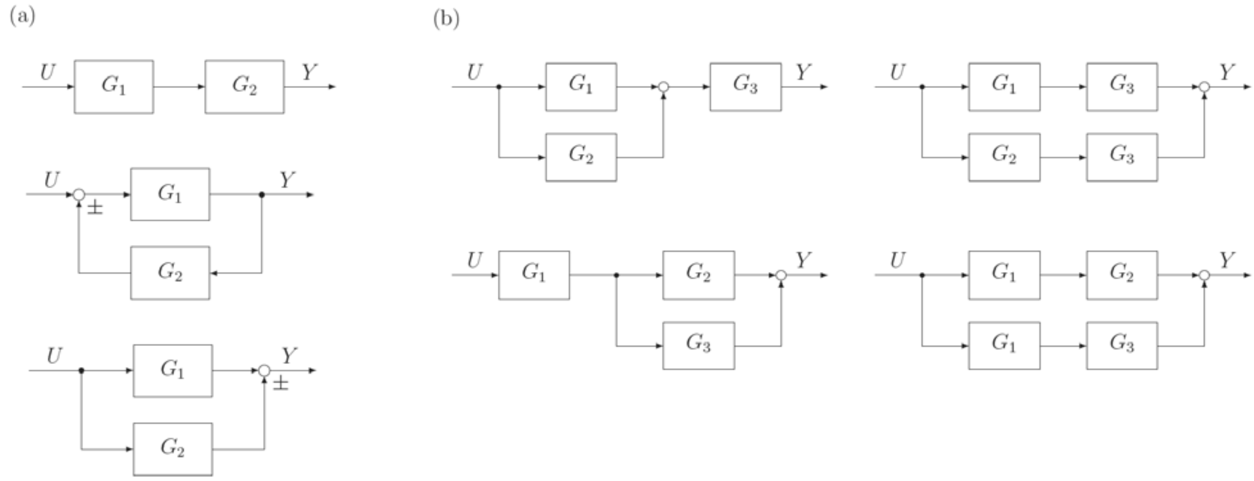

Figure 3: Illustration of Block Circuits: (a) Basic Topologies, (b) Algebra of Transfer Functions.

It is evident that this series are results from the sampling process, following denoted as $(f_n)_{n\in\mathbb{N}}$.

1. If the series $(f_n)_{n\in\mathbb{N}}$ satisfy the inequality
$$|f_n| \leq M\gamma^n, \qquad \forall n \in \mathbb{N}$$
and for suitable $\gamma, M > 0$, (f_n) is called from exponential order.

2. $S(\mathbb{N},\mathbb{R})$ denotes the set of all series which are from exponential order.

3. With
$$\mathscr{Z}(f_n)(z) = \sum_{n=0}^{\infty} f_n z^{-n}$$
a mapping $\mathscr{Z}$ is defined, the so-called z–Transform.

Due to the correlation of the sampling process the z–Transform correspond with the Laplacian. Assume

$$\tilde{f}(t) = \sum_{n=0}^{\infty} f(nT)\delta(t-nT)$$

for $T > 0$ and $\delta\colon\ \mathbb{R} \to \mathbb{R}\cup\{\infty\}$, defined by

$$\delta(t) = \begin{cases} \infty & \text{for } t = 0, \\ 0 & \text{otherwise,} \end{cases}$$

and the condition $\int_{\mathbb{R}} \delta(t)\,\mathrm{d}t = 1$.

Apply on $\tilde{f}$ the Laplacian it results in

$$\mathscr{L}(\tilde{f})(s) = \int_0^{\infty} \sum_{n=0}^{\infty} f(nT)\delta(t-nT)\mathrm{e}^{-st}\,\mathrm{d}t = \\ = \sum_{n=0}^{\infty} f(nT)\mathrm{e}^{-snT}.$$

In summary for $f_n = f(nT)$ and $z = \mathrm{e}^{sT}$ it results the formula of the z–Transform.

This relation between the two domains allows the definition of a sampling system by observing a continuous LTI-system with a discretisation and reconstruction interface as illustrated in Figure 4.

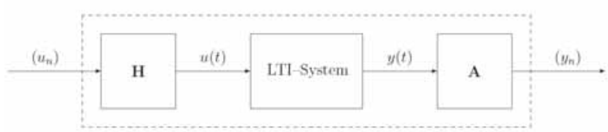

Figure 4: Split-up of a sampling system with sample and hold interface and a corresponding continuous LTI-system.

3.2 Nonlinear systems

Nonlinear Systems has as well a description which is oriented to an input-output relation. It is a generalisation of the state space description for linear time invariant systems, given by two mappings $f\colon \mathbb{R}\times\mathbb{R}^n\times\mathbb{R}^m \to \mathbb{R}^n$ and $g\colon \mathbb{R}\times\mathbb{R}^n\times\mathbb{R}^m \to \mathbb{R}^p$ for $m,n,p \in \mathbb{N}$ satisfying the form

$$\begin{aligned} x'(t) &= f\left(t,x(t),u(t)\right), \\ y(t) &= g\left(t,x(t),u(t)\right). \end{aligned} \tag{11}$$

Not only this systems are addressed when the term nonlinear is used. Also systems with a certain saturation or discretisation effect are covered in this field. Systems like Rate-Limiter, Saturation, Quantizer and Hit Crossing are counting to this systems as well. Nevertheless the general mathematical formulation fits also to this class of systems.

4 Simulation Models in SIMULINK

The introduced mathematical framework for SIMULINK is suitable to describe systems in a formal way. For simulation aspects this description is not helpful or offers optimizing opportunities. But in a modelling context this framework offers a new level in the coexistence of modelling and simulation. The common approach is that the modelling process provides a description of the model and the simulation environment runs the calculation.

The description of the model is normally given in a mathematical framework or an computer science approach. This produce the first problem, there is no common layer where the model can be compared or analysed. The introduced framework offers the possibility to separate a abstract mathematical model, graphical simulation model and the model implemented in the simulation environment by itself. In case of SIMULINK models this approach offers an abstract layer for model descriptions.

References

[1] Angermann A., Beuschel M., Rau M., Wohlfarth U. *MATLAB–SIMULINK–Stateflow*. 6th Edition. Oldenburg; 2009.

[2] Scherf H. *Modellbildung und Simulation dynamischer Systeme*. 4th Edition. Oldenburg; 2010.

[3] Körner, A. *Analyse und Simulation dynamischer Systeme in Simulink mit Web-Interface* [master thesis]. [Insitute for Analysis and Scientific Computing, (abrev. State)]. Vienna University of Technology; 2012.