An Approximate Differentiation Method of Inverse Simulation based on a Continuous System Simulation Approach

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Abstract. Inverse simulation techniques allow inverse solutions to be found for a range of problems involving dynamic systems described by sets of linear or nonlinear ordinary differential equations. Techniques in common use generally involve iterative solutions based on discretised descriptions but continuous system simulation tools can also provide solutions and are often simpler to apply and computationally more efficient. This paper presents a method of inverse simulation which involves use of a very simple, but effective, approximation for derivative terms within the model. Discussion of results for linear and non-linear examples leads to the conclusion that this technique can be applied to a wide range of dynamic models that are of practical importance for engineering applications.

Introduction

Conventional modelling and simulation involves a process of finding a model 'output' for a given set of initial conditions and time history of 'inputs', whereas inverse modelling and simulation is a process in which 'inputs' are found that will produce prescribed model 'outputs'. Inverse simulation methods provide an alter-native to the use of mathematical techniques for the inversion of dynamic models, particularly in the nonlinear case. Not only do they avoid the complexities of the mathematical approaches for nonlinear models but they also provide insight that may not otherwise be so readily available from conventional simulation techniques.

A number of established inverse simulation methods involve discretisation of continuous models. Examples include the so-called differentiation approach [1], [2], the widely-used integration based approaches [3], [4] and optimisation-based methods [5], [6]. A paper by Thomson and Bradley provides a useful review [7] of some of these techniques, which are essentially iterative in nature and were developed, initially, for aeronautical applications.

Other techniques are based on continuous system simulation models and include the use of differential algebraic equation (DAE) solvers such as those available in the Modelica® environment (see, e.g. [8]) but DAE methods do not yet appear to have been applied routinely to large and complex models of the type that typically arise in engineering applications. A useful and proven alternative involves the use of feedback principles and continuous system simulation tools (see, e.g. [9-13]).

The approximate differentiation method of inverse simulation outlined in this paper provides yet another approach which involves continuous system simulation methods and may be simpler to apply in some cases. In common with other methods of inverse simulation it has obvious limitations, but can be used for a range of model structures of importance for engineering applications.

1 The Approximate Differentiation Method

This is a continuous simulation equivalent of the discrete 'differentiation method'. The basic idea is to rearrange the given model in state space form so that the inputs of interest appear on the left hand side of the equations.

Derivatives of state variables appearing on the right hand side can then be approximated using a simple continuous representation based on the use of an integrator block and feedback pathway, as shown in the block diagram of Figure 1. This may be seen from the first order equation defining that system which is given by:

$$\frac{dw}{dt} = \frac{1}{T}\left(v(t) - w(t)\right) \quad (1)$$

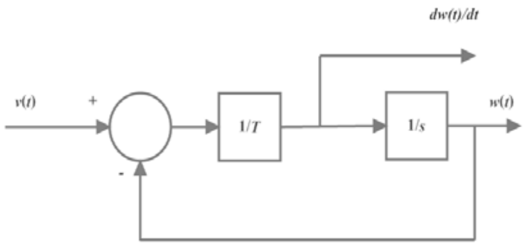

Figure 1: Block diagram of the approximate differentiator.The block 1/*s* represents the operation of integration in terms of the Laplace variable *s*.

If the variable *v*(*t*) in Figure 1 is replaced by $x_{id}(t)$ which represents the desired time history for a state variable x_i then, provided the time constant *T* is very small in relation to the dynamics of the given model, the variable *w*(*t*) in Figure 1 is a close approximation to the desired variable $x_{id}(t)$ and to the state variable $x_i(t)$. The quantity found at the input to the integrator block in Figure 1 is given by $\frac{1}{T}(x_{id}(t) - x_i(t))$ and is thus an approximation to the derivative $\dot{x}_i$. Hence a derivative of a state variable $\dot{x}_i$ within a given state-space model may be replaced by a quantity $\frac{1}{T}(x_{id}(t) - x_i(t))$ where $x_{id}(t)$ is the desired time history.

The approach, which was mentioned in an invited keynote lecture at the 8th EUROSIM Congress in September 2013 [14], is best presented through a simple illustrative example which has also been used in investigations of other inverse simulation methods [6], [15]. Consider a linear single-input single-output system model of the form:

$$\dot{x} = Ax + Bu \quad (2)$$
$$y = Cx + Du \quad (3)$$

where $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix}$, $B = \begin{bmatrix} 1 \\ -5 \\ 69 \end{bmatrix}$,

$$C = [1 \quad 0 \quad 0]\,,\ D = 0$$

Simple linear analysis shows that this linear single-input single-output (SISO) system model has poles at positions *s* = -1 rad/s, *s* = -2 rad/s and *s* = -3 rad/s and zeros at *s* = -0.5000±j7.0534 rad/s. The range of frequencies of interest for this model is from 0 to 30 rad/s. The set of ordinary differential equations for the system as given above is:

$$\dot{x}_1 = x_2 + u \quad (4)$$
$$\dot{x}_2 = x_3 - 5u \quad (5)$$
$$\dot{x}_3 = -6x_1 - 11x_2 - 6x_3 + 69u \quad (6)$$
$$y = x_1 \quad (7)$$

Let the desired output be denoted by x_{1d}. The derivative $\dot{x}_1$ may then be approximated by $\frac{1}{T}(x_{1d} - x_1)$ and we have a new 'output equation' of the form:

$$u = x_2 - \dot{x}_1 = x_2 - \frac{1}{T}(x_{1d} - x_1) \quad (8)$$

We now have a modified set of equations of the form:

$$\dot{x}_1 = -\frac{1}{T}x_1 + \frac{1}{T}x_{1d} \quad (9)$$
$$\dot{x}_2 = \frac{5}{T}x_1 + 5x_2 + x_3 - \frac{5}{T}x_{1d} \quad (10)$$
$$\dot{x}_3 = -(6 + \frac{69}{T})x_1 - 80x_2 - 6x_3 + \frac{69}{T}x_{1d} \quad (11)$$
$$u = -\frac{1}{T}x_1 - x_2 + \frac{1}{T}x_{1d} \quad (12)$$

This set of equations has zeros at *s* = -1 rad/s, *s* = -2 rad/s, *s* = -3 rad/s and poles at *s*=-0.5 ± *j*7.0534 rad/s and at s= -1/T rad/s so, clearly, the zeros of the inverse simulation model are at the same positions as the poles of the given model and the poles lie at the positions of the zeros of that model, apart from an additional pole at s= -1/T. Provided the time constant *T* can be made very small, this additional pole of the inverse simulation can be positioned at a point in the *s*-plane far from the other poles and zeros, where its effect is insignificant. For example, a value of *T* of 0.001s would give an additional pole at *s* = -1000 rad/s, and this is so far removed from all the other poles and zeros that it would have a negligible influence on the dynamic behaviour of the inverse simulation. One could, of course, make the time constant *T* even smaller but this would tend to increase the stiffness of the inverse simulation and there is a clear trade-off between the overall accuracy of the inverse simulation and computational efficiency.

Consider the specific case of an desired time history for the state variable $x_1(t)$ which takes the form of a triangular waveform involving a negative going ramp, starting from zero at time *t* = 0 with a gradient -0.5 units/s, changing to a positive slope of 0.5 units/s at time *t* = 1.5s and then repeating this pattern at time *t* = 3 s.

Using MATLAB® software to implement the inverse simulation based on Equations (9)-(12) for this desired form of output waveform and a value of T of 0.001s, we find that the required input has the form shown in the upper trace of Figure 2. The lower traces show the required output, together with the output found from a forward simulation for the given model when the waveform obtained from the inverse simulation is applied as input. It can be seen from this that the two plots almost coincide, so the input found from inverse simulation generates the required output almost exactly for the chosen value of T.

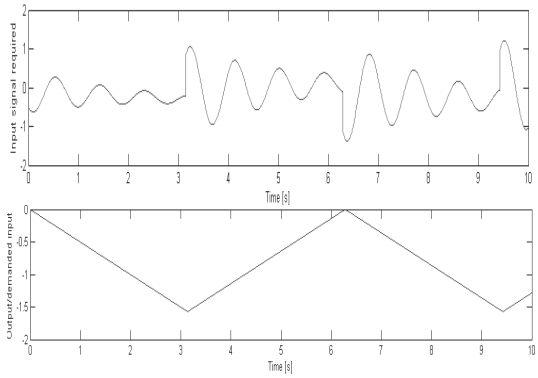

Figure 2: The upper trace shows the input found from inverse simulation of the linear SISO system of Equations (2) and (3). The lower traces (superimposed) show the demanded output together with the output obtained from application of the input found from inverse simulation to the conventional forward simulation model for this system.

2 An Example Involving a Nonlinear Model

Consider a mathematical model of a coupled-tanks system shown in schematic form in Figure 3, involving two interconnected tanks of liquid, (Tank 1 and Tank 2) with input flow rates, Q_{i1} and Q_{i2} respectively. These inputs are from electrically driven variable-speed pumps. There is a single outlet flow Q_{23} from the second tank which may be adjusted manually by means of a tap. Both tanks are equipped with sensors that can detect the level of liquid and provide a proportional output as an electrical voltage signal.

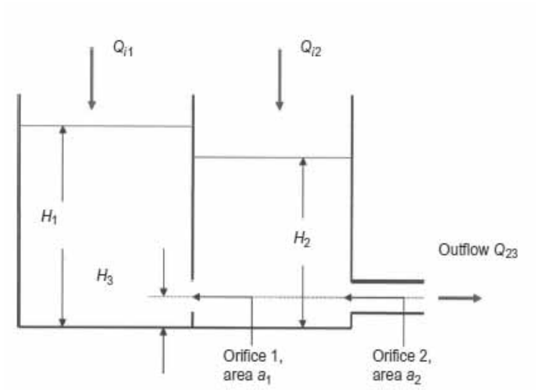

Figure 3: Schematic diagram of the coupled-tanks system.

If the levels of liquid in the two tanks (H_1 and H_2) are regarded as output quantities and the flow rates from the pumps as inputs, a two-input two-output second-order nonlinear state-space description may be developed for this system using simple physical principles based on the fact that the rates of change of volume of liquid in each tank must be equal to the difference between the total flow rate into that tank and the total flow rate out (see e.g. [13], [16]).

For situations in which the liquid level in Tank 1 is greater than the level in Tank 2 the equations are as follows:

$$\frac{dH_1}{dt} = \frac{Q_{i1}(t)}{A_1} - \frac{C_{d1}a_1}{A_1}\sqrt{2g(H_1(t) - H_2(t))} \tag{13}$$

$$\frac{dH_2}{dt} = \frac{Q_{i2}(t)}{A_2} + \frac{C_{d1}a_1}{A_2}\sqrt{2g(H_1(t) - H_2(t))} - \frac{C_{d2}a_2}{A_2}\sqrt{2g(H_2(t) - H_3)} \tag{14}$$

Values of parameters are defined below for a specific laboratory-scale system [13], [16]:

Cross-sectional areas of tanks $A_1 = A_2 = 9.7\times10^{-3}$ m^2;
Cross-sectional area of orifice 1 $a_1 = 3.956\times10^{-5}$ m^2;
Cross-sectional area of orifice 2 $a_2 = 3.85\times10^{-5}$ m^2;
Coefficient of discharge of orifice 1 $C_{d1} = 0.63$;
Coefficient of discharge of orifice 2 $C_{d2} = 0.58$;
Height of outlet above base of tank $H_3 = 0.03$m;
Gravitational constant $g = 9.81$ m/s^2;
Maximum flow rate $Q_{i1max} = Q_{i2max} = 5\times10^{-5}$ m^3/s;
Minimum flow rate $Q_{i1min} = Q_{i2min} = 0$ m^3/s;
Maximum liquid level $H_{1max} = H_{2max} = 0.3$m.

This model is nonlinear in structure because of the nonlinear relationship between the liquid levels in the Tanks 1 and 2 and the flow between them and also because of the nonlinear relationship between the output flow rate and the liquid level in Tank 2.

The inverse simulation developed from the application of the approximate differentiation method is readily, obtained using the principles outlined above, and involves the following set of equations:

$$\frac{dH_1}{dt} = \frac{1}{T}\left[H_{1req}(t) - H_1(t)\right] \quad (15)$$

$$\frac{dH_2}{dt} = \frac{1}{T}\left[H_{2req}(t) - H_2(t)\right] \quad (16)$$

$$Q_{i1}(t) = \frac{A_1}{T}\left[H_{1req}(t) - H_1(t)\right] + C_{d1}a_1\sqrt{2g(H_1(t) - H_2(t))} \quad (17)$$

$$Q_{i2}(t) = \frac{A_2}{T}\left[H_{2req}(t) - H_2(t)\right] - C_{d1}a_1\sqrt{2g(H_1(t) - H_2(t)} - C_{d2}a_2\sqrt{2g(H_2(t) - H_3)} \quad (18)$$

where $H_{1req}(t)$ and $H_{2req}(t)$ are the required time histories of liquid levels in Tanks 1 and 2 respectively.

Figure 5 and Figure 6 show results obtained using MATLAB® for the method outlined above for a case involving the required time histories of level changes shown in Figure 4. The value of the time constant T used in the equations for the inverse simulation is 0.1s, which is very small in relation to the dynamic characteristics of the system. The results from the inverse simulation for the desired level changes of Figure 4 are given in Figure 5 and these show that the pattern of input flow rate changes do not cause either of the inputs to reach the maximum flow rate of 5×10^{-5} m^3/s or to drop to zero flow rate (the minimum allowed) at any time in the simulated test. The results do, however, show strong interactions between the two tanks, especially during the initial phase of the test where the input flow to Tank 2 falls rapidly to compensate for the rising level in Tank 1 (and thus allows the required level in Tank 2 to be maintained). When the input flow patterns shown in Figure 5 are applied to a forward simulation of the coupled-tanks system, the levels of liquid in the two tanks are very close to the required levels in Figure 4, as is shown by the results of Figure 6 which show maximum differences of the order of 1.4×10^{-17} m between the levels found from forward simulation results and the desired levels.

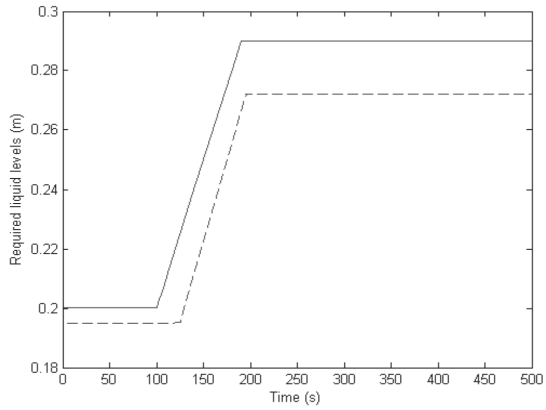

Figure 4: Required liquid levels for the first case considered.

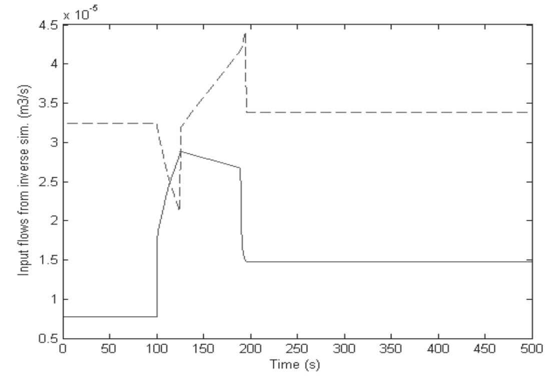

Figure 5: Flow rates determined by inverse simulation for the first case considered.

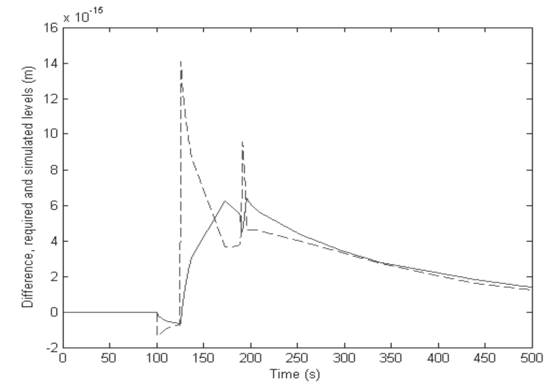

Figure 6: Differences between reference liquid level time histories and liquid levels found by applying inputs from inverse simulation to the forward simulation model.

The pattern of demanded reference levels shown in Figure 7 relates to a case in which the required level changes result in one of the input flow rates reaching its maximum value of 5×10^{-5} m^3/s. The input flow rate to Tank 1 remains at that maximum value for a period of about 40s and then falls to a new constant value of about 4×10^{-5} m^3/s, while the input flow to Tank 2 drops to zero, as is shown in Figure 8. In this case, if the input flow rates obtained from inverse simulation are applied to the forward simulation model, significant differences are found between the levels achieved and the desired levels, as shown in Figure 9. The inverse simulation thus shows, in this case, that for the given system configuration the required pattern of liquid level changes cannot be achieved. Hence, this might suggest that design changes would be required within the system if this pattern of level changes was an essential requirement for some specific application.

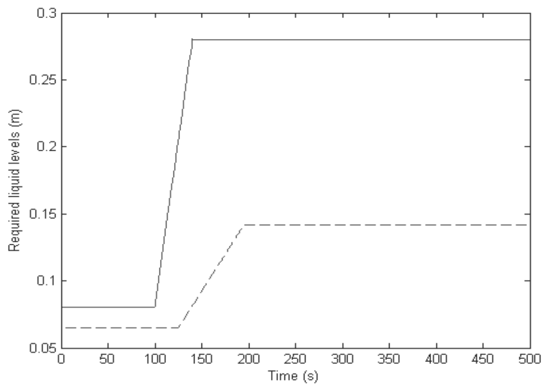

Figure 7: Required liquid levels for second case involving a larger difference between final levels in Tank 1 and Tank2 resulting in a required flow rate for Tank 1 that exceeds the maximum.

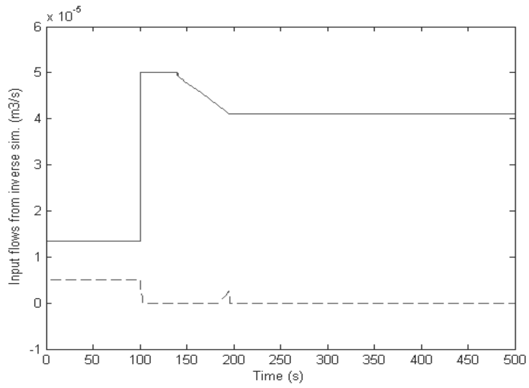

Figure 8: Input flow rates found using inverse simulation for level variations defined in Figure 7.

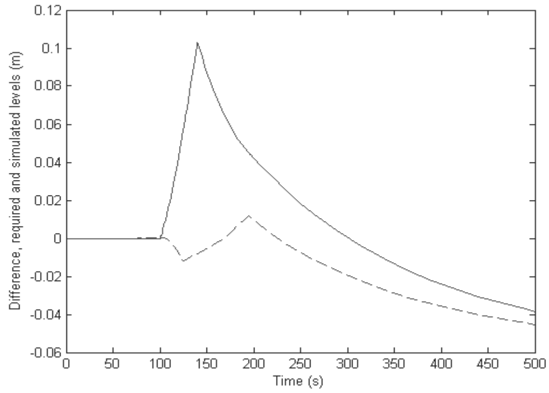

Figure 9: Differences between the desired levels and the levels found from forward simulation of the coupled-tanks model for the case defined by the inputs of Figure 8.

3 Discussion and Conclusions

The linear example involving a single-input single-output model defined in state-variable form shows that the approximate differentiation approach to inverse simulation can give results in which poles of the inverse simulation match the zeros of the given model. The approximation is equivalent to adding one pole at a point in the *s*-plane which is located on the negative real axis far from all the poles and zeros of the model. The effect of this additional pole resulting from the approximation can thus be made negligibly small by choosing a small value for the time constant associated with the differentiation process.

In the case of a multi-input system, if all the inputs are to be found by inverse simulation, the number of additional time constants would be equal to the number of inputs, but the time constants would again have negligible effect if they had appropriate small values.

The success of the approximate differentiation approach for the case of a model involving nonlinear equations has been demonstrated through use of the coupled-tanks example. In this case, provided the model input variables are within their limiting values, the inverse simulation gives an accurate prediction of the inputs required to produce desired model output time histories.

The method also provides the user with a clear indication of the effects of input limits when they arise, thus providing insight about the reasons why a specific desired time history of outputs is not achievable. Depending on the context in which inverse simulation is being applied, such situations may lead to modification of the desired pattern of outputs or to a change in the design of the system represented by the model.

As mentioned in the introductory section of this note, another commonly-used approach to inverse simulation which uses continuous system simulation methods is based upon feedback techniques (see e.g. [13]). This has been found to be a powerful approach and has been applied to a wide range of practical systems. However, that method requires the design of a feedback structure around the given simulation model, which can be time-consuming and difficult for those with little experience of closed-loop system design. It can also present significant problems if limit cycle oscillations arise. The approximate differentiation method thus provides an interesting alternative approach which avoids such difficulties but is also based on continuous system simulation principles.

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