Randomisation and Grading of Complex Questions in the E-Learning System Maple T.A.

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Abstract. At Vienna University of Technology the Maple T.A. (abbr. for Testing and Assessment) e-learning system was introduced into mathematical courses. It proved to be a useful e-learning addition to usual teaching methods. Students are able to train their calculation skills and deepen their knowledge in different mathematical disciplines outside of the lectures. Further Maple T.A. is a helpful tool to test and assess the skills of students, since it contains an assignment module and a gradebook module that provide these tasks. During the two years of usage of Maple T.A. at the Vienna University of Technology a large amount of questions and exercises were created for different mathematical topics. But with increasing complexity of the topics the creation of questions became more and more difficult. The main problems were the creation of randomly specified questions and the final grading of these questions. To overcome these challenges two Maple libraries were developed. One of them is responsible for the randomization of more complex objects and the other one aims to simplify the grading process for these more complex objects in Maple directly instead within the web environment.

Introduction

At the Vienna University of Technology the web-based e-learning Maple T.A. is used to assist the assessment of students' mathematical skills. It proved to be a useful addition to the common way of teaching math.

Former studies [1, 2] showed that with the possibility to practice their mathematical skills, students were able to deepen their knowledge in different mathematical disciplines outside of the lectures. Due to the randomization abilities of Maple T.A. students always get randomized specifications of questions and examples, and so they can practice the same tasks over and over again, but with differing parameters. Further the software additionally provides an assignment and a gradebook module which are helpful tools to perform assignments for students' skills.

During the last two years of usage of Maple T.A. at the Vienna University of Technology a large amount of questions and examples were created for different mathematical topics. But with increasing complexity of the topics the creation of questions became more and more difficult. Where for the beginner's lectures the common functions of Maple T.A. satisfied the demands adequately, difficulties arose in continuative lectures. In particular the randomization and grading of more complex mathematical objects (e.g. matrices, vectors, etc.) with special properties were challenges that had to be overcome.

For this purpose two additional Maple libraries were developed and added to the Maple core of the Maple T.A. system. One of them is responsible for the randomization of these mathematical objects and the other one aims to simplify the grading process for them.

1 Randomisation Library

1.1 Function Definitions

Function FromSet. Random elements from a set are selected. The data types of the set elements are not restricted. According to the parameters set the return value of the function is either a vector or a single value.
Parameters:

- *Set* – set with selectable values
- *Count* – number of elements to be selected
- *Distinct* – with or without replacement
- *Sorted* – determines whether the return vector is sorted or not

Function MakeInts/FromInts. For given interval boundaries the function returns a sequence of either a specified number or all integers between these boundaries.
Parameters:

- *Min* – lower boundary of the interval
- *Max* – upper boundary of the interval
- *Count* – number of desired random values

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Function MakeRats/FromRats. The function is similar to MakeInts/FromInts. The return values are here rational numbers that are restricted by some additional parameters for the denominators.
Paramters:
- *Min* – lower boundary of the interval
- *Max* – upper boundary of the interval
- *Count* – number of desired random values
- *MinDenom* – smallest possible denominator
- *MaxDenom* – largest possible denominator

Function Vec. For a given dimension this function returns a random vector with user-defined entries.
Paramters:
- *Set* – set of numbers for random vector entries
- *Dim* – desired dimension of the vector
- *Zerocount* – number of zeros in the vector

Function VecInts. A random vector with integer entries is created and returned.
Paramters:
- *Dim* – dimension of the return vector
- *Max* – for each entry k: $k \leq k_{\max}$
- *Zerocount* – number of zeros in the vector

Function Mat. The Mat-function generates random matrices. It is possible to define the matrix dimension as well as a desired shape.
Paramters:
- *Set* – set of numbers for matrix entries
- *rows* – number of matrix rows
- *cols* – number of matrix columns
- *zerocount* – number of entries equal zero
- *shape* – determines the matrix shape (e.g. triangular, symmetric, etc.)

Function MatInts. Just like the vector functions, MatInts creates matrices only with integer entries.
Paramters:
- *Max* – absolute values of entries are ≤ *Max*
- *rank* – desired rank of the matrix
- *rows* – number of matrix rows
- *cols* – number of matrix columns
- *zerocount* – number of matrix entries equal zero

Function MatIntsDef. This function delivers a random symmetric matrix with integer entries and specifies numbers of positive (pos), negative (neg) and eigenvalues equal zero (zero). The dimension of the matrix is defined by: dim = pos + neg + zero, and the rank equals pos + neg.
Paramters:
- *Max* - absolute values of entries are ≤ *Max*
- *pos* - number of positive eigenvalues
- *neg* - number of negative eigenvalues
- *zero* - number of eigenvalues equal zero
- *zerocount* - number of entries equal zero

1.2 Algorithms Example

To demonstrate the functionality of one randomization algorithm, we give a short example for the creation of a random $m \times n$ matrix. Consider the following function call:

```
MatInts(Max=4,rows=7,cols=5,rank=3)
```

We start with creating a matrix that obviously has the desired rank r. Therefore we choose r random integer values within the given range and write them in the "diagonal" of the first r rows.

$$\begin{pmatrix} -4 & 0 & 0 & 0 & 0 \\ 0 & -4 & 0 & 0 & 0 \\ 0 & 0 & -4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{1}$$

Then we progressively fill the first r rows with random integer entries within the given range. When doing that, we have to make sure that we do not reduce the rank of the matrix. It can be shown, that for each selected entry there is at most one number $\alpha \in \mathbb{Q}$, that makes the matrix rows linear dependent. So if α lies in the given range, we have to select another value.

$$\begin{pmatrix} -4 & -4 & 4 & 1 & -3 \\ 0 & -4 & -2 & 0 & -4 \\ 2 & 4 & -4 & -4 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{2}$$

Now that we have a matrix with random entries and the desired rank we proceed with finding $m - r$ linear dependent vectors with integer entries within the given range and replace the last $m - r$ rows. For this purpose we first transpose the matrix and build a base of the span of the first r columns, which is easier to handle.

$$B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 3/2 & -7/12 & 7/6 \\ 2 & 1/4 & 3/2 \end{pmatrix} \quad (3)$$

This transformation ensures that each linear combination of the columns with integer linear coefficients $|\alpha_i| \leq Max$ has correct entries in the first r components. After that we calculate the common denominator (q) of all entries and multiply the last $m - r$ rows with q.

$$q = 12, \; P = \begin{pmatrix} 18 & -7 & 14 \\ 24 & 3 & 18 \end{pmatrix} \quad (4)$$

We apply Gauss elimination modulo q.

$$Q = \begin{pmatrix} 6 & 5 & 2 \\ 0 & 3 & 6 \end{pmatrix} \quad (5)$$

At last we randomly choose a vector $v \in \{-Max, \ldots, Max\}^r$ such that $Qv = 0$ and the absolute values of all entries of the vector $w = Bv$ are less or equal `Max`, for example:

$$v = \begin{pmatrix} 2 \\ -4 \\ -2 \end{pmatrix}, \; w = \begin{pmatrix} 2 \\ -4 \\ -2 \\ 3 \\ 0 \end{pmatrix} \quad (6)$$

We transpose w and insert it into the matrix

$$\begin{pmatrix} -4 & -4 & 4 & 1 & -3 \\ 0 & -4 & -2 & 0 & -4 \\ 2 & 4 & -4 & -4 & 1 \\ 2 & -4 & -2 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad (7)$$

After repeating the last step for the remaining rows and a final permutation of rows and columns we get the resulting matrix of our algorithm:

$$\begin{pmatrix} -2 & -1 & 3 & 2 & 2 \\ -4 & -2 & 3 & 0 & 2 \\ 2 & 1 & -3 & -2 & -2 \\ -4 & -2 & 3 & 0 & 2 \\ -4 & 4 & 1 & -3 & -4 \\ -4 & -2 & 0 & -4 & 0 \\ 4 & -4 & -4 & -1 & -2 \end{pmatrix} \quad (8)$$

This function is also used when generating random definite matrices. Consider the function call `MatIntsDef(M,pos,neg,zero)`. So the aim of the algorithm is to get a matrix with pos positive eigenvalues, neg negative eigenvalues, and eigenvalue zero with mulitplicity zero. Therefore we calculate a random matrix $B \in Z^{(\text{pos+neg})\times(\text{pos+neg+zero})}$ with the rank `pos+neg` using the `MatInts` function. Let then the first pos rows of B be B^+ and the last neg rows of B be B^-.

We determine the matrix $A = (B^+)^{\mathrm{T}}B^+ - (B^-)^{\mathrm{T}}B^-$. Regarding to Sylvester's law of inertia [4] A has `pos` positive, `neg` negative eigenvalues and eigenvalue zero with multiplicity `zero`. This algorithm is repeated until the absolute value of all entries of A are less or equal M.

2 Library Grading

Automatic grading of answers is not always an easy task. It is necessary to have well defined rules when evaluating the correctness of entered sets, vectors or matrices. For this purpose Maple T.A. provides the Maple-Graded question type that connects to the Maple engine when grading student's responses. The question designers have to write some Maple code that is responsible for the handling of the answers. According to the complexity of the question this task can be quite tricky and may need long development time. With the additional Maple grading library, that takes care of most of these problems, this time can be reduced. Following functions are supported:

- *Function Expr.* This function is for common mathematical expressions. They are checked for mathematical equivalence to the given correct answer. For this object type no partial grading is possible. So the student's response is either completely correct or wrong.
- *Function ExprDiff.* ExprDiff is used when grading anti-derivatives. The response is derivated and then compared to the original specification of the question.
- *Function Set.* For sets partial grading seems reasonable. Students may enter subsets or supersets of the correct ones. To handle this issue a return value is calculated that lies between 0 and 1, depending on how correct the student's answer is.
- *Function Vec.* The Vector grading function returns 0, if the entered vector and the correct one have different lengths. Otherwise the return value is determined by dividing the correct entries by the total number of vector entries.
- *Function Mat.* Evaluating the correctness of matrices follows the same way as for vectors. If the response matrix and the correct matrix have different sizes the entered matrix is graded with 0. In all other cases the return value is again the correct number of entries di-

vided by the total number of entries.

- *Function FundSys.* This function is used for grading real fundamental systems which define the sets of solutions of systems of homogeneous linear differential equations. The order of the differential equation determines the cardinality of the set. So the return value is calculated similar to the one of the set-grading function: The order of the differential equation is equal $|C|$, $|R \backslash C|$ corresponds to the greatest number of linear independent real solutions in the response and $|R \backslash C|$ is equal to the smallest number of functions, that have to be removed from the response set, to get a set of linear independent real solutions.

3 Usage Experience

The usage of the new libraries simplified the creation of Maple T.A. questions significantly. Due to the fact, that most of the algorithms used in questions were assumed by the new libraries and the Maple engine, the exercises became less error-prone and better readable for non-editors. On the other hand the increased use of Maple entailed some other unexpected problems. With many requests at the same time a certain slowdown of the system could be noticed. Also the creation of algorithms within questions has to be done with slight more attention. Infinite loops or other computationally intensive operations can cause very long computing times for the system. To avoid these problems question development within a local Maple installation is advisable.

Nevertheless the Maple T.A. system became more reliable with the use of the new libraries. The needed time for the development of exercises decreased in a noticeable way. So a larger amount of high-quality questions could be placed at disposal for the students.

4 Summary and Outlook

The recent work with the new developed library gained acceptance of the system among both students and question designers. But as everywhere there is still plenty of room for improvement and further development of new algorithms. According to the above mentioned system speed concerns the existing algorithms should also go under closer investigations. Further we are looking forward to future releases of new Maple T.A. versions, since some helpful features for question development have not been integrated into the system yet. Maybe the most wanted one is the conjunction of students' responses. With it the creation of high quality questions would be easier.

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