Mathematics and Eggs – Do Those Two Terms have Something in Common?

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Abstract. Modelling of objects of daily life is a topic which could be a motivating and fascinating access to mathematics education. For this reason the paper shows a possibility, how an object which is common to all students in school, the egg, can be an item for an exciting discussion in schools. Based on two mathematical definitions a figure is constructed. Based on those the description in polar-coordinates and Cartesian implicit equations is developed. This stepwise modelling cycle is shows the way how modelling of daily objects could be done in school and demands of curricula could be implemented.

Introduction

The concept of modelling for education has been discussed for a long time. It is a basic concept in all parts of sciences and in particular in mathematics. This concept is a well accepted fundamental idea (see [14]), in case of the preliminary-definition of Schweiger [12] is used:

A fundamental idea is a bundle of activities, strategies or techniques, which
1. *can be shown in the historical development of mathematics,*
2. *are sustainable to order curricular concepts vertically,*
3. *are ideas for the question, what is mathematics, for communicating about mathematics*
4. *allows mathematical education to be more flexible and clear*
5. *have a corresponding linguistic or activity-based archetype in language and thinking.*

Therefore it is not remarkable that the concept of modelling can be found in a lot of different curricula all over the world. For example in the Austrian curriculum for mathematics in grammar-schools you can find a lot of quotations for it [7].

By interpreting those quotations in the curriculum modelling can be seen as a process-related competence. That means:

- translating the area or the situation to be modelled in mathematical ideas, structures and relations
- working in a given or constructed mathematical model
- interpreting and testing results

Those process-related competencies have been described by many people, e.g. Pollak [10], Müller and Wittmann [8], Schupp [13], Blum [1]. Regarding to all the developments in modelling Blum and Leiß [2] has designed a modelling cycle which is designed by a more cognitive point of view.

1 Problems in Realm of Students' Experiences in Mathematics Education

Problems of real-life, like problems in environment, sports or traffic, are often a starting point for calculations and applications of mathematics. But before using mathematics in such fields it is necessary that the problem is well understood. This asks for a lot of time and dedication, because it is necessary to trans-late the problem from reality to mathematics and back to reality. Therefore models are used as an adequate description of the given situation. Modelling through problems in realm of students' experiences means creating an image of reality which allows describing complex procedures in a common way. Creating such an image has to observe two directions as Krauthausen [4] quotes:

- Using knowledge for developing mathematical ideas.
- Developing knowledge about reality by its reli-ance on mathematics.

If such problems are discussed in mathematics education it will be possible that students are more motivated for mathematics. But there are a lot of other arguments why such problems should be discussed. They

- help students to understand and to cope with situations in their everyday life and in environment,

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- help students to achieve necessary qualifications, like translating from reality to mathematics,
- help students to get a clear and straight picture of mathematics, so that they are able to recognize that this subject is necessary for living, and
- motivate students to think about mathematics in a deep going way, so that they can recall impor-tant concepts even if they were taught a long time ago.

If a teacher is concerning the listed points, then he will be able to find a lot of interesting topics which he/she is allowed to discuss with students. Exemplarily I want to show the motivation problems in realm of students' experiences by observing an egg.

2 The Egg

If we have a closer look at an egg, we will see that its shape is very harmonic and impressive. Considering a hen's egg it is obvious that the shape of all those eggs is the same. Because of the fascinating shape of eggs I tried to think about a method to describe the shape of such an egg with mathematical methods. Searching the literature I found some material from Münger [9], Schmidt [11], Malina [6], Wieleitner [17], Loria [5], Timmerding [16] and Hortsch [3]. The book of Hortsch is a very interesting summary about the most important results of 'egg-curves'. He also finds a new way for describing egg-curves by experimenting with known parts of 'egg-curves'. The modality how the authors are getting 'egg-curves' is very fascinating. But none of them has thought about a way to create a curve by using elementary mathematical methods. The way how the authors describe such curves are not suitable for mathematics education in schools. So I thought about a way to find such curves with the help of well known concepts in education. My first starting point is a quotation of Hortsch [3]: 'The located ovals were the (astonishing) results of analytical-geometrical problems inside of circles.' The second point of origin are the definition of 'egg-curves' found by Schmidt [11] and presented in Hortsch [3].

2.1 First Definition (Schmidt [11])

Schmidt quotes: 'An 'egg-curve' can be found as the geometrical position of the base point of all perpendiculars to secants, cut from the intersection-points of the abscissa with the bisectrix, which divide (obtuse) angles between the secants and parallel lines in the intersec-

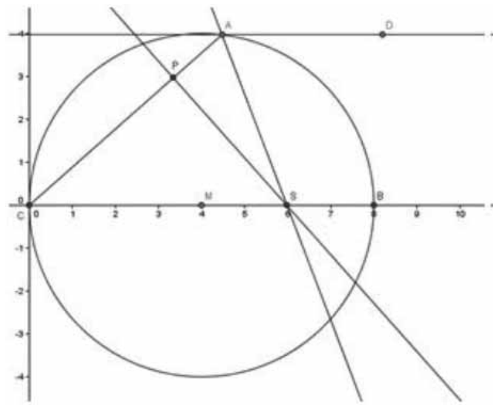

Figure 1. Translating the definition of Schmidt to the DGS

tion-points of secants with the circle circumference in halves. The calculated formula is $r = 2a\cos^2\varphi$ or $(x^2+y^2)^3 = 4a^2x^4$.'

In education the role of education is gaining in importance. Different systems, like computer-algebra-systems (CAS), dynamical-geometry-software (DGS) or spreadsheets, are used in education. With the help of technology it is possible to design a picture of the given definition immediately. In the first part I use a DGS because with its help it is possible to draw a dynamical picture of the given definition. The DGS I am using is GeoGebra. It is free of charge and very suitable in education because of its handling.

First of all we construe the one point P of such an egg as it is given in the definition, as shown in Figure 1.

According to the construction instruction I have first construed a circle (center and radius arbitrarily), then a secant from C to A (points arbitrarily). After that I have drawn a parallel line to the x-axis through the point A—

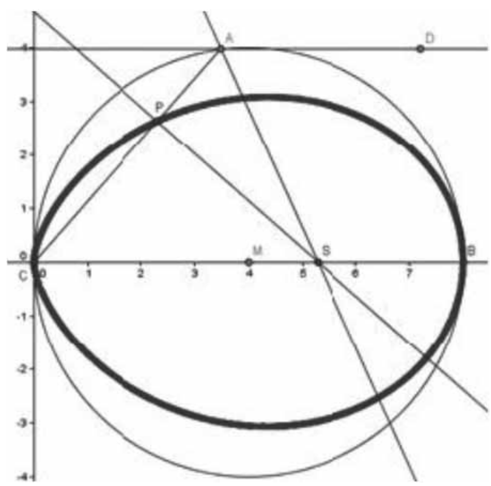

Figure 2. Egg-curve construed by DGS

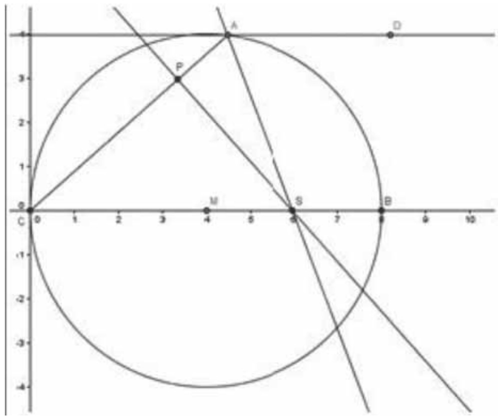

Figure 3. Initial situation for calculating the equations of the egg-curve.

the intersection point of secant and circle—and determined the bisecting line CAD, which is cut with the x-axis. So we get point S. Now we draw the perpendicular to the secant through S. The intersection point of the secant and the perpendicular is called P and is a point of the 'egg-curve'. Now I activate the 'Trace on' function and use the dynamical aspect of the construction. By moving A towards the circle the 'egg-curve' is drawn as Schmidt has described it. This can be seen in Figure 2.

Now we have to find a way to calculate the formulas $r = 2a\cos^2\varphi$ and $(x^2+y^2)^3 = 4a^2x^4$ as mentioned above.

Let us start with Figure 3: We know, because of the construction that the triangle CPS is right-angled. Furthermore we can recognize that the distance CP and PS is the same and that the triangle CAB is also rectangular, because it is situated in a semicircle. This can be seen in the following figure, where I have also drawn the real 'egg-curve' as dashed line (Figure 4).

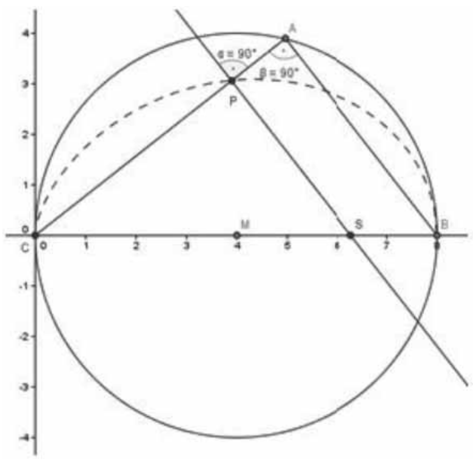

Figure 4. Affinity of the triangles

Because of the position of the points C (0, 0), A (x, y), B ($2r$, 0) and the construction instruction the coordinates of point S and P can be calculated. Therefore only a little bit of vector analysis is necessary. The calculation can be done in the CAS Mathematica.

First of all I have to define the points and the direction vector of the bisecting line w:

```
{c = {0, 0}, a = {x, y}};
a - c
```

$\{x, y\}$

```
1 / Norm[a - c] (a - c)
```

$$\left\{\frac{x}{\sqrt{\mathrm{Abs}[x]^2 + \mathrm{Abs}[y]^2}}, \frac{y}{\sqrt{\mathrm{Abs}[x]^2 + \mathrm{Abs}[y]^2}}\right\}$$

```
w = -1 / √((x)^2 + y^2) {x, y} + {1, 0}
```

$$\left\{1 - \frac{x}{\sqrt{x^2+y^2}}, -\frac{y}{\sqrt{x^2+y^2}}\right\}$$

$$w = \left\{1 - \frac{x}{\sqrt{x^2+y^2}}, -\frac{y}{\sqrt{x^2+y^2}}\right\};$$

Now I can calculate the equation of the normal form of the bisection line, cut it with the x-axis and define the intersection-point S.

$$wn = \left\{y, \sqrt{(x)^2+y^2} - x\right\};$$

```
wn.{u, v} == wn.a
```

$$u\,y + v\left(-x + \sqrt{x^2+y^2}\right) == x\,y + y\left(-x - \sqrt{x^2+y^2}\right)$$

```
wn.{u, v} == wn.a /. v → 0
```

$$u\,y == x\,y + y\left(-x + \sqrt{x^2+y^2}\right)$$

```
First[%] / y == Last[%] / y
```

$$u == \frac{x\,y + y\left(-x + \sqrt{x^2+y^2}\right)}{y}$$

```
Simplify[%]
```

$$\sqrt{x^2+y^2} == u$$

$$s = \left\{\sqrt{x^2+y^2}, 0\right\};$$

Now I can calculate the intersection point P of the secant and the perpendicular through S.

$$\mathrm{Solve}\left[\left\{x\,u + y\,v == (x)\left(\sqrt{(x)^2+y^2}\right), -y\,u + (x)\,v == 0\right\}, \{u, v\}\right]$$

$$\left\{\left\{u \to \frac{x^2}{\sqrt{x^2+y^2}}, v \to \frac{x\,y}{\sqrt{x^2+y^2}}\right\}\right\}$$

$$p = \left\{\frac{x^2}{\sqrt{x^2+y^2}}, \frac{x\,y}{\sqrt{x^2+y^2}}\right\};$$

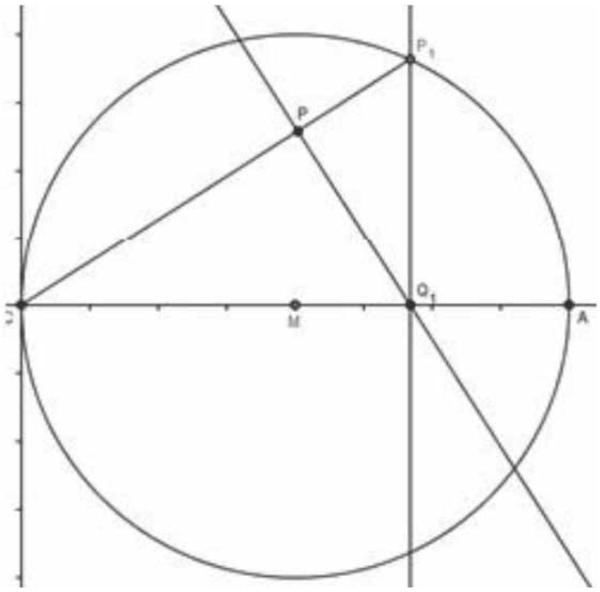

Figure 5. Translating the definition of Münger to the DGS

Now all important parts for finding the 'egg-curve' are calculated. Let us have a closer look at figure 4. It is easy to recognize that there are two similar triangles – triangle CPS and triangle CAB. The distance CP shall be called r and the radius of the circle shall be called a. The distance CB has now the length $2a$. The other two distances which are needed CA and CB has the length $\sqrt{x^2+y^2}$. Now we can apply the similarity of the triangles:

$$\frac{CA}{r}=\frac{2a}{CS} \quad \Leftrightarrow \quad \frac{\sqrt{x^2+y^2}}{r}=\frac{2a}{\sqrt{x^2+y^2}} \tag{1}$$

Transforming this equation, delivers:

$$x^2+y^2=2ar$$

Now I use the characteristic of the right-angled triangle CAB and call the angle ACB φ. For the cosine of this angle I get:

$$\cos\varphi=\frac{\sqrt{x^2+y^2}}{2a} \quad \Leftrightarrow \quad 4a^2\cos^2\varphi=x^2+y^2$$

By inserting this connection in the equation above I get:

$$4a^2\cos^2\varphi=2ar$$

Shortening this equation, delivers:

$$r=2\,\mathrm{a}\cos^2\varphi$$

By substituting r and $\cos\varphi$ it is possible to get the implicit Cartesian form, mentioned in the definition:

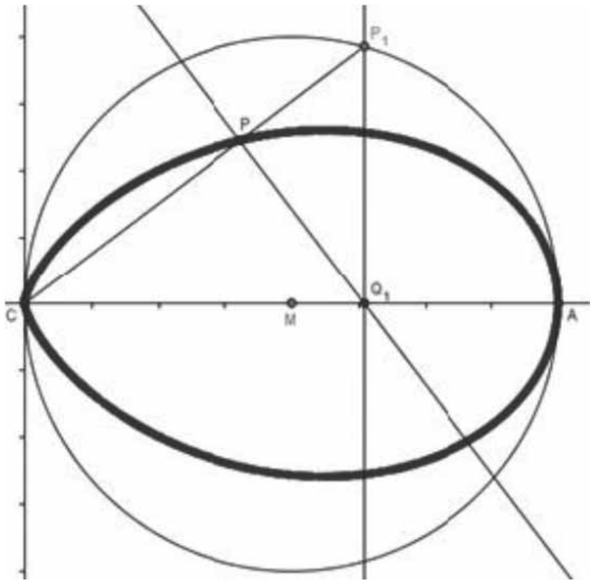

Figure 6. Egg-curve construed by DGS

$$\sqrt{x^2+y^2}=2\,\mathrm{a}\cos^2\varphi$$

$$x^2+y^2=4a^2\frac{x^4}{r^4}=4a^2\frac{x^4}{(x^2+y^2)^2}$$

respectively

$$(x^2+y^2)^3=4a^2x^4$$

As we have seen the 'egg-curve' has been modelled by elementary mathematical methods. Through using technology teachers and students get the chance to explore such calculations by using the pivotal of modelling. Through such calculations the necessity of polar-coordinates can get obvious.

2.2 Second Definition (Münger [9])

Another construction instruction is formulated by Münger [9]. He quotes: 'Given is a circle with radius a and a point C on the circumference. CP_1 is an arbitrarily position vector, P_1Q_1 the perpendicular to the x-axis, Q_1P the perpendicular to the vector. While rotating the position vector around C point P is describing an egg-curve. The equation of this curve is $r=a\cos^2\varphi$, in Cartesian form $(x^2+y^2)^3=4a^2x^4$.'

As it is given in the construction instruction a circle (radius arbitrarily) and a point C on the circumference of the circle is constructed. Then we construe an arbitrarily point P_1 on the circumference of the circle. The perpendicular to the x-axis is construed through P_1 which is cut with the x-axis and delivers Q_1. After that the perpendicular to the secant CP_1 through Q_1 is construed. All these facts can be seen in Figure 5.

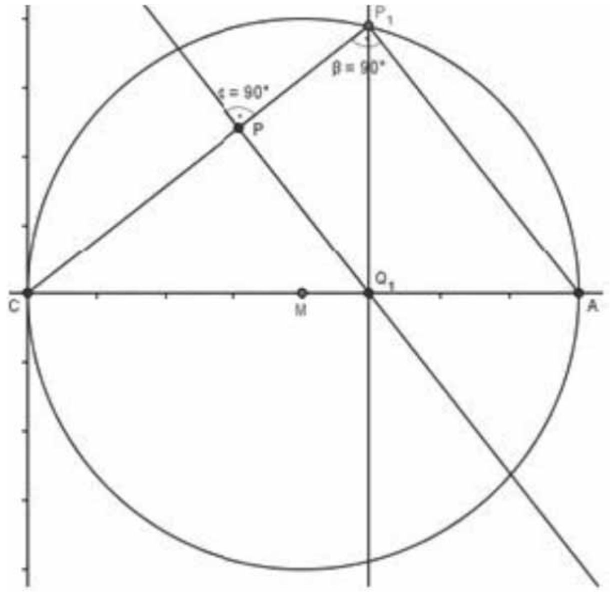

Figure 7. Affinity of the triangles

If the point P_1 is moved toward the circle, P will move along the 'egg-curve'. It will be easier to see if the 'Trace on' option is activated (see Figure 6).

The formula given by Münger can be found in a similar way as the other formula was found. The most important fact which has to be seen here is that in this picture two rectangular triangles CPQ_1 and CP_1A exist. Those triangles are similar (see Figure 7).

The coordinates of the points can be found mentally – without any calculation:

$$C\ (0,0),\ \ P_1(x,y),\ \ A\ (2a,0),\ \ Q_1\ (x,0)$$

If the distance CP is called r, then the coordinates of P will not be used. Otherwise they can be calculated analytically. For the sake of completeness I write down the coordinates of P:

$$P\left(\frac{xy^2}{x^2+y^2},\frac{y^3}{x^2+y^2}\right)$$

If we use the similarity of the both triangles, then the following equation will be obvious:

$$\frac{\sqrt{x^2+y^2}}{r}=\frac{2a}{x}$$

Through elementary transformation, because of the mathematical fact that $\cos\varphi=\sqrt{x^2+y^2}/r$ in the triangle CP_1A and substitution of the term $\sqrt{x^2+y^2}$ by $2a\cos\varphi$ the following equation is calculated:

$$2a\,x\cos\varphi=2a\,r$$

The result is:

$$r=x\cos\varphi$$

Because of the fact (assumption in the calculation) that x is part of our circle—it is the x-coordinate of P_1—, x can be substituted by $x=a\cos\varphi$, where a is the radius of the starting circle. So the formula of Münger is found with polar coordinates:

$$r=a\cos^2\varphi$$

If the implicit cartesian form should be stated, another substitution has to be done. The result is:

$$(x^2+y^2)^3=4a^2x^4.$$

3 Epilogue

Looking at objects of daily life through the lense of mathematics can show interesting and motivating (mathematical) results. The grade of complexity is not an essential attribute for discussing objects of realm of students' experiences. It is necessary to show students that mathematics allocates methods and instruments to analyse objects in daily life. For this reason Austrian and German mathematics educators founded the ISTRON group in 1990. This group has the aim to look at problems in daily life and to show teachers how they can improve their education by implementing such examples. Lots of materials can be found in volume 0 – 11. A more detailed look at mathematical aspects of an egg can be found in the article of Siller, Maaß, Fuchs [15] which is accepted for the next volume of ISTRON.

All in all it is necessary that adequate problems are shown for education. Problems which are already known in research should be adapted respectively constructed for education. The fields for educational research in this area is should be expanded and strengthened.

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