Decision Making with a Random Walk in a Discrete Time Markov Chain

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The paper describes a Markov model for multi-criteria and multi-person decision making. The motivation results from a demand observed in the early stages of an innovation process. Here, many alternatives need to be evaluated by several decision makers with respect to several criteria. The model derivation and description can be split into the evaluation process and the decision process. The pair wise comparisons can be combined by weighting them according to the importance of the criteria and decision makers, resulting in a discrete-time Markov chain. A random walk on this DTMC models the decision process, where a longer state sojourn time implies a better alternative. We believe that this model meets the demands in the early stages of an innovation process.

1 Description of the problem

We consider the problem of evaluating alternatives in the early stages of an innovation process. In this application area alternatives need to be evaluated by several decision makers with respect to different criteria. There are possibly many alternatives that need to be considered; therefore it is necessary to make the evaluation process fast and simple. The problem parameters are the following:

- A possibly large number of alternatives may be involved
- Several decision makers may be involved
- Several evaluation criteria (both quantifiable and soft) may be involved
- The evaluation criteria may be weighted according to relevance
- The opinions of the decision makers may be weighted according to expertise
- Little or no information is available about the alternatives; the decision makers base their evaluations on intuition or guesswork
- The decision makers have to decide fast due to the possibly large number of alternatives

The following example describes the intended application. During an innovation workshop a large number of ideas are produced. Each idea is described only with a title and a short characterisation. An innovation team must identify the top ten ideas to bring them forward to the first stage of a stage-gate process [4]. Little or no quantifiable information is available about the ideas, therefore it is not possible to rank the ideas based on objective criteria. Instead, only subjective impressions are available at this stage, enabling decisions of the form “A is better than B” with respect to a given criterion. Each member of the team might have a different area of expertise or competence, and to each of them can be assigned a different weight for different criteria. For each pair wise comparison the decision maker and the criteria is noted.

The following questions need to be addressed: How to model an evaluation process and a decision process with the specified parameters? How to deal with inconsistent or non-transitive evaluations, which can occur due to the subjective nature of the evaluations? How to determine the top alternatives?

2 State of the art

In the field of multi-criteria decision making (MCDM) many methods have been developed for specialised applications. Detailed information about MCDM can be found, for example in [10], [8] and [1]. Thirty available methods are discussed in [7]. Two more general methods which can be used in the early stages of an innovation process are AHP (the Analytic Hierarchy Process) [2, 11] and cost-benefit analysis (CBA) [3].

However, AHP and CBA are not directly applicable to the intended application. AHP does not support multiple decision makers, unless additional aggregation strategies are applied to merge the individual evaluation result [9, 13, 6].

Furthermore, AHP requires consistent transitive evaluations in order to compute a valid result. CBA needs measurable and quantifiable criteria to compute a valid result.

However, inconsistent evaluations and soft criteria are often present in the early stages of an innovation process. Accordingly, AHP and CBA cannot be the preferred methods to evaluate alternatives under these circumstances.

3 A DTMC-based model for evaluation and decision process

This section describes how to model both the evaluation process and the decision process. The evaluation process is similar that in AHP but with only one level of "is better than". In the given application it is not applicable to use more than one level of difference, because there is no strict differentiation possible to get more detailed decisions for two reasons. Firstly, little or no information about the alternatives may be available and secondly the use of soft criteria.

The model of the decision process is described using an analogue situation.

3.1 Evaluation process

Based on the assumption that little or no information about the alternatives is available, the evaluation process is implemented as pair wise comparisons between all alternatives, concerning all criteria. These comparisons only ask for a decision of the following form "is better than". This solution allows comparisons according to non-measurable evaluation criteria, such as taste or preference.

We assign weights to the decision makers according to their expertise and to the evaluation criteria according to their relevance, and scale these weights to sum up to one. We build a weighted directed graph, where each comparison adds an edge from the less preferred alternative to the better one. The edge weight results from the weight of the criterion and the decision maker. After adding the edges corresponding to the comparisons to the graph, we can scale the edge weights such that the sum of all outgoing edges of a node is one. The resulting graph is a discrete-time Markov chain, where the edges lead from the less preferred to the better alternatives.

A detailed description of the mathematics behind the evaluation process can be found in [4].

In the evaluation process we have participants p_k with $k = 1 \dots K$, criteria c_l with $l = 1 \dots L$ and alternatives am with $m = 1 \dots M$. Each participant p_k makes pair wise comparisons between two alternatives a_{m1} and a_{m2} with respect to criterion c_l. We denote this as follows: $p_k(c_l)\colon a_{m1} > a_{m2}$.

Next we need the coefficient α_{kl} to assign weights to each evaluation made by the participants. Each coefficient α_{kl} contains information about the relevance of participant p_k with respect to criterion c_l and describes the importance of criteria c_l where larger values imply greater importance. In the matrix A of dimension $K \times L$ we store the coefficients that satisfy

$$0 \le \alpha_{kl} \le 1 \text{ and } \sum_{k=1}^{K} \sum_{l=1}^{L} \alpha_{kl} = 1.$$

Each evaluation of the participant p_k with respect to criterion c_l is represented in the matrix E_{kl} of dimension $M \times M$. We build the matrix as follows:

$$E_{kl(m_1,m_2)} = \begin{cases} 1/(\delta_{m1} + 1) & p_k(c_l)\colon a_{m2} > a_{m1} \\ 0 & \text{otherwise} \end{cases},$$

where δ_{m1} represents the number of non-zero entries in the m_1-th row of matrix E_{kl}. The main diagonal coefficients are set as follows:

$$E_{kl(m_1,m_1)} = 1 - \sum_{m_2=1, m_2 \ne m1}^{M} E_{kl(m_1,m_2)}.$$

Finally, we need a matrix P of dimension $M \times M$. P contains the complete set of evaluations and is computed from the weighted sum of all E_{kl}. P is also a stochastic matrix and computed by

$$P = \sum_{k=1}^{K} \sum_{l=1}^{L} \alpha_{kl} E_{kl}.$$

The result of the evaluation process is a DTMC containing all evaluations of the participants as weighted edges.

3.2 Decision process

We believe that after having built this DTMC from the decision makers' evaluations, the decision process corresponds to a random walk on this resulting Markov chain. We assume that the sojourn time of better alternatives is larger than for inferior ones. To determine the preferred alternatives, the DTMC is solved, resulting in the steady state probability vector. The more incoming edges with large weights one node has, the more comparisons were made preferring that alternative. The more outgoing edges with large weights one node has, the fewer comparisons preferred that alternative.

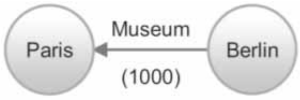

Figure 1. 1000 students evaluated Paris better than Berlin for the museums.

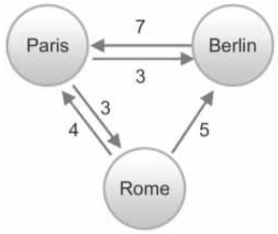

Figure 2. Example bus departure frequencies (x times a week)

Consequently the better alternatives with more incoming edges have a larger probability in the result. The resulting probability vector is then interpreted as a ranking of the alternatives, where larger probabilities show a higher rank.

To obtain the best alternative we take a look to an analogue situation: Assumed all students from a university meet after summer and evaluate impressions from their holiday trips in Europe with respect to several criteria. As a result they designed a visitor guide from statements like "I've visited Berlin for its museum and then I've visited Paris. Paris was even better than Berlin." This result could be visualized like shown in Figure 1.

Furthermore we assume a bus company with special travel offers for students. This company may use the visitor guide to adjust the connection frequencies between the capitals in Europe. In our example (see Figure 2), the connection frequency from Berlin to Paris is higher than the other way, if Paris gets more positive evaluations in comparison with Berlin (all criteria merged):

Finally we have a student who makes a tour in Europe. In every capital he takes the first bus which is departing to another capital on his arrival at the bus station. Now, some friends of the student want to join his tour. In which capital do they have the highest possibility to meet the student?

Because the student decides on each bus station randomly, we believe the student acts like a "random walker" in a directed graph. With the different bus departure frequencies the solution is to meet the student in the capital with the highest sojourn time of the random walker.

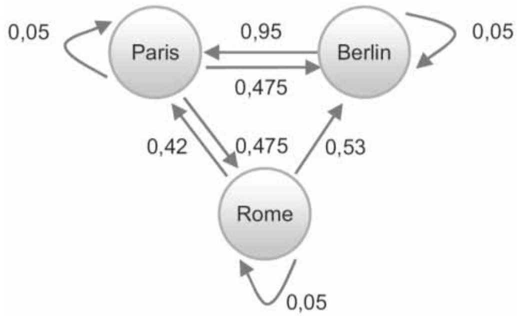

Figure 3. DTMC with normalized weights (corresponding to the departure frequency)

Since the sojourn time of a random walker corresponds to the values in a steady state probability vector of a discrete-time Markov chain (DTMC) his friends should build and solve a DTMC. DTMCs are well-researched mathematical models with many applications in Science and Engineering. A DTMC is described by a stochastic matrix P and a probability vector π. The steady-state solution of the DTMC contains the probabilities of each of the system states and is given by the solution of the linear system of equations $\pi P = \pi$. Markov chains are drawn as weighted directed graphs, where the nodes represent the states and the edges represents the possible state transitions. The weights associated with the edges describe the one-step probabilities for each state transition. A state or set of states of a Markov chain is called absorbing, if it contains only incoming edges [12].

The capitals are represented as nodes and the bus connections as directed edges. The weight of each edge corresponds to the departure frequency of the buses for each connection (Figure 3). Figure 3 also contains nodes with self-pointing edges. A self-pointing edge with value 1 means, no bus will depart from this capital (consequently the student will stay there). Self-pointing edges with values lower than one represent the probability for the student to spend another day in the capital (in this example, the value is assumed to be five per cent for each capital).

Before computing the ranking vector π, we need to consider one case that distinguishes our approach from a genuine random walker: The student decides to pick a capital randomly and not to decide using the bus station. In this case we assume the probability to travel to a certain capital is equal for all capitals (see Figure 4). To solve this problem we build a new matrix F of dimension $M \times M$ with interconnections to all capitals with the weights $1/n$:

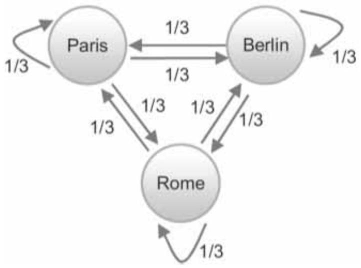

Figure 4. Graph representing matrix F with the weight $1/n$ (n: number of capitals)

Now we can compute matrix $R = (1-\varepsilon)P + \varepsilon F$ using the parameter ε with $0 < \varepsilon \ll 1$ which describes the probability that student picks the next capital at random not using the buses. The matrix R is stochastic and irreducible. Furthermore R defines a Markov chain without absorbing states (Figure 5).

The algorithm to compute the ranking vector π contains both, the evaluation process and the decision process. The algorithm is given by a sequence of five steps:

1. Choose the coefficients α_{kl}.
2. Choose the value for ε.
3. Enter the values for the pair wise comparisons a_{m1} and a_{m2} into E_{kl} and calculate the values for the main diagonal of E_{kl}.
4. Compute R by adding all E_{kl} weighted by the coefficients.
5. Solve the DTMC, computing the steady state solution π.

After termination of the algorithm the result of each alternative a_m is equivalent to the value π_m. The larger the value π_m is, the higher is the rank of a_m.

Starting with the initial probabilities $\pi_0 = (1/3\,,1/3\,,1/3)$ and $\varepsilon = 0.01$ the solution vector of the example is $\pi = (\text{Paris,Berlin, Rome}) = (0.44, 0.34, 0.22)$.

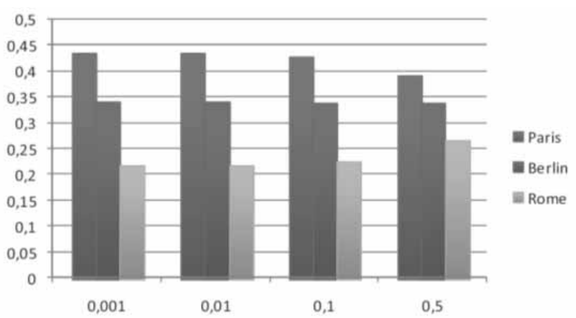

Figure 6. Influence of ε on the values in the solution vector π.

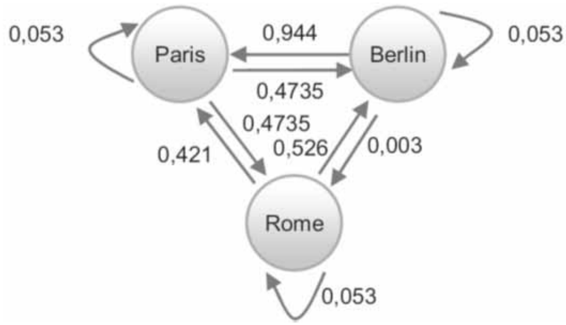

Figure 5. DTMC from Matrix R with $\varepsilon = 0.01$

Consequently the friends of the travelling student have the highest possibility to meet the student in Paris.

Even though ε changes the values for π_m, this influence can be neglected as experiments have shown. A large value of ε only dampens the vector solutions, but does not influence the ranking positions. Experiments have shown that the larger the value of ε, the less the gap between the values in π (see Figure 6).

The same approach we illustrated for the travelling student, we chose to determine the ranking of alternatives in the early stages of an innovation process. Therefore we built a Markov chain R based on the weighted pair wise comparisons made by the participants. Afterwards we send the random walker through the Markov chain who wants to visit all preferred alternatives. To avoid to be caught in a local maximum he can choose his next randomly, not using recommendations. Therefore we use the matrix F with entries of size $1/M$. The parameter ε now contains the probability that the random walker decides to continue his journey on a randomly chosen alternative. After computing matrix R, we compute the sojourn time of a random walker for each alternative and received the solution vector π.

Finally, the steady state solution of the DTMC then yields the ranking of the alternatives.

4 Discussion of the advantages in our approach

We believe that using our DTMC based decision process we can solve the following problems. The top alternatives are identified by their larger values in the probability vector, which correspond to the sojourn time of a random walker.

A DTMC can also handle contradicting and intransitive comparison results; it can contain edges running in opposite directions. By using the weights assigned to the decision makers and criteria to weight the edges, we can easily combine their comparisons made into one DTMC. The steady state solution of the DTMC then yields the ranking of the alternatives.

Furthermore, our approach has the following advantages. The method allows quantifiable and soft evaluation criteria. The method enables decision makers to evaluate alternatives with little or no information available. The method combines a decision making problem with an established mathematical method (discrete-time Markov chains). By combining only some of the comparisons made, it is possible to compute intermediate results during the evaluation process. The method is simple to use, because we only ask for "better than" decisions. The method can easily represent levels of competence of the decision makers with respect to the criteria.

We will now discuss all of these points one by one and show why they apply to our method.

The model can handle contradicting and intransitive comparisons. This statement we can confirm because a DTMC can handle contradicting and intransitive comparison results; it can contain edges running in opposite directions. This case can occur when evaluations concerning different criteria are made. E.g.: Car 1 is cheaper than car 2 and car 2 is more comfortable than car 1. This will lead to edges in opposite directions in the DTMC. However the weight of the edges will be different depending on the weights assigned to each of the criteria.

The model can handle inconsistent evaluations. This statement can also be confirmed, because a DTMC can handle these evaluations. Two inconsistent evaluations can occur, when decision makers have contradicting opinions concerning soft criteria. For example: Expert 1 thinks, car 1 is more comfortable than car 2 because of a better driving seat. Expert 2 prefers car 2 over car 1, concerning comfort, because he likes the more spacious backseat. This again results in two edges running in opposite directions, but having different weights according to the decision makers' expertise.

The model can handle soft and quantifiable evaluation criteria. This statement can be confirmed as well. "Better than" comparisons allow soft as well as measurable evaluation criteria.

The advantage of this simple decision is that they can be done much easier within a short period of time. "Wrong" decisions should be cancelled out by the mass of other evaluators, criteria and transitive relationships.

The model can handle multiple decision makers. To achieve that, we add up weighted decision matrices of each decision maker. The more we value the opinion of a decision maker, the more weight we can give his evaluations. This results in more expertise or more influence in the decision process. Therefore this statement was confirmed as well.

Intermediate results during the evaluation process are available. We can confirm this statement, because adding one decision of a decision maker keeps the properties of a DTMC intact. Therefore it can be solved having added only some of the edges corresponding to the evaluations. The remainder of the probability of one node stays in that node itself. This implies the initialization of all the matrices to the Identity matrix. These intermediate results allow us to stop the evaluation process before all pair wise comparisons have been made, and getting a ranking. The quality of that ranking has to be determined otherwise (discussed in Section 5).

In some of our experiments we observed the so-called rank reversal effect. One case where it occurs is, when we insert a new node similar to an existent node (e.g. two equal cars in different colours). In this case the new rank of both nodes can be smaller than the original node's rank. This occurs because the incoming edges probabilities are divided between the similar nodes. In our opinion, when adding or removing alternatives and their respective evaluations, we change the nature of the problem. We found an analogue case which can be explained by the following example: The ranking of the German Football League is also obtained by pair wise comparisons, the games. Assuming, we remove Hoffenheim and all the games they played from the evaluations, the following phenomena can occur. Teams that won against Hoffenheim loose points and teams that lost against Hoffenheimdo not loose points. This can change the ranking. When removing a contestant without removing their games or other teams' points, the global ranking is not affected. For a DTMC this removal would mean adding edges which correspond to evaluations that were not made, but that were implicitly there because of transitivity.

Since we do not want to add evaluations, which were not actually entered by a decision maker, rank reversal is a property of our model.

5 Conclusion and outlook

In this paper we described a model for the decision process in the early stages of an innovation process. We used pair wise comparisons and weights for decision makers and criteria to combine them to form a discrete-time Markov chain. A random walk on this chain models the actual decision process. The solution of the Markov chain yields the probability vector, which gives us a ranking of the alternatives. In contrast to existing methods, our model can easily handle inconsistent evaluations, soft criteria and multiple decision makers.

Since intermediate rankings can also be computed, we see improvement potential in the model by reducing the number of comparisons necessary to reach a certain goal. As far as our experience goes the most interesting goals in the early stages of an innovation process are the following:

- Identify the best alternative: Due to limited resources or project type only the best alternative is needed.
- Identify the top x alternatives: This comes directly from the properties of the stage gate process in innovation management. Limited resources restrict the number to only some innovation projects. Here the ranking of these top x among each other is unimportant.

In these cases the evaluation process can be aborted if the evaluation goal is reached. To be able to implement this, we need to find heuristics to decide which comparison should be asked for next. This might involve ordering the possible judgements according to their effect on the ranking vector. Then, the algorithm can prompt the decision makers to input the more influential judgements first. Our assumption is that the more judgements are made, the less effect on the ranking vector is measurable. That means that the probability of a rank exchange decreases. As first experiments showed, the number of necessary judgements to reach one of these evaluation goals decreases considerable in comparison to obtaining an accurate ranking result by entering all possible pairwise comparisons.

We think that our method can be applied to many other applications with equal conditions, also beyond the innovation management.

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