SVD-AORA Model Order Reduction Method for Large-Scale Dynamic Linear Time Invariant System

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Abstract. In this paper we present a joint model for order reduction for dynamic linear time invariant (LTI) system, which we call SVD-AORA (Singular Value Decomposition-Adaptive Order Rational Arnoldi). The SVD-AORA method is an extension of the SVD-Krylov based method. It is based on linear projection using two projection matrices (V and Z). The first matrix V is generated using the Krylov technique through the AORA method, the second matrix Z is generated using the SVD technique by the resolution of the Lyaponuv equation. After the resolution of the Lyaponuv equation, the solution obtained (The gramian observability matrix g_o) is decomposed using the SVD technique and thus we obtain the second projection matrix Z. The use of the AORA method enhances the numerical efficiency thanks to its relative lower computation complexity and the use of the SVD technique preserves the stability of the reduced system. The proposed method gives a reduced order model asymptotically stable, captures the essential dynamics of the original model and minimizes the absolute error between the original and the reduced one. The results of the proposed method are compared with other popular approach of order reduction in the literature which is the SVD-Krylov method. The reduced systems obtained by the proposed method have better performance compared to SVD-Krylov method. The method is explained through two numerical systems of different order.

Introduction

Technological world, physical and artificial processes are mainly written by mathematical models which can be used for simulation or for control. Among these models the LTI of high order. However, these high order models are difficult to manipulate and analyze because of the fact that the resolution of these models is indeed very demanding in computational resources, storage space, and mainely in CPU time. Hence the necessity of model order rection technique.. In the literature there exist different reduction methods of linear time invariant system (Arnoldi, Lanczos [1, 2, 3, 4],Rational Arnoldi [5],Rational Lanczos [6, 7], AORA [8], AOGRA [9], AORL [10], PRIMA [11],...); which performance differantly. Among these performances we can mention:

- A significantly reduced number of variables or states (required for description of a given model) compared to the original model,
- The simulation should be quick and does not require large memory space,
- The computational complexity associated with the evaluation of the reduced model should be significantly lower than the original model,
- Stability of reduced model must be guaranteed,
- Minimization of error between the original model and reduced one.

To Bring this performances, we depict in this paper the SVD-AORA method. This paper in organized as follow: in section 3, basic tools are developed. In section 4, a description of SVD-AORA method is given with application in theoretical models. In section 5, a comparative study is presented. Section 6 concludes the work.

1 Preliminary

This section reviews some basic mathematical tools related to the linear dynamical system.

1.1 Moment matching

Let a state space representation of linear dynamical system be as [12, 13]:

$$\sum = \begin{cases} \frac{dx(t)}{dt} = Ax(t) + Bu(t) \\ y(t) = Cx(t) + Du(t) \end{cases} \tag{1}$$

The transfer function of linear system described as equation 1 is given by [14, 15]:

$$F(s) = C(sE - A)^{-1}B \tag{2}$$

If $F(s)$ is expanded as a power series around a given finite point $s_0 \in \mathbb{R}$, then we obtain [9, 8]:

$$F(s) = f_0 + f_1(s - s_0) + f_2(s - s_0)^2 + \ldots + f_{n-1}(s - s_0)^{n-1} \tag{3}$$

Where, n is the order of original system and the $f_k,\ for\ k = 0 : n-1$ coefficients present the moment matching of the dynamical linear system around the frequency s_0. The f_k coefficients are described by [10, 5]:

$$f_k(s_0) = C(s_0E - A)^{-(k+1)}B \tag{4}$$

1.2 Krylov Subspace

Let a frequency s_i be for $i = 1 : \hat{i}$, a matrix $\psi = (A - s_iE)^{-1}E$ and a vector $\xi = (s_iE - A)^{-1}B$. Then the Krylov subspace is obtained by [1, 2, 16, 17]:

$$\mathbb{K}(\psi, \xi) = \{\xi, \psi\xi, \ldots, \psi^{n-1}\xi\} \tag{5}$$

1.3 H_∞ error

Take a linear asymptotically stable system as in 1. The H_∞ norm is computed by this relation [18, 1, 19]:

$$H_\infty = sup_{w \in \mathbb{R}} \|F(jw)\|_2 \tag{6}$$

The reduced transfer function obtained by the use of model order reduction method is $\hat{F}(s) = \hat{C}(s\hat{E} - \hat{A})\hat{B}$. Then, the H_∞-norm error between the original system and reduced one is determined by the following relationship:

$$\| F - \hat{F} \|_{H_\infty} = sup_{w \in \mathbb{R}} \| f(jw) - \hat{f}(jw) \|_2 \tag{7}$$

2 SVD-AORA Model Order Reduction Method

The accuracy and the computational efficiency of the AROA and SVD-Krylov methods still insufficient in term of H_∞error minimization and the stability preservation of the reduced system. In this section we give a main mathematical problem formulation. Also, we present the main steps of the proposed method SVD-AORA and the results obtained by the use of two models.

2.1 Mathematical problem formulation

The mathematical problem consists on determining the state space parameters (order $k << n$) of the reduced model $\hat{\sum}$ from the state space parameters (order n) of the original model $\sum$ [1, 20, 12, 21] by using the proposed model order reduction method:

$$\sum = \begin{cases} \frac{dx(t)}{dt} = Ax(t) + Bu(t) \\ y(t) = Cx(t) + Du(t) \end{cases} \tag{8}$$

in which $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times p}$, $C \in \mathbb{R}^{p\times n}$ and for simplicity we take $D = 0$, we obtain.

$$\hat{\sum} = \begin{cases} \frac{d\hat{x}(t)}{dt} = \hat{A}\hat{x}(t) + \hat{B}u(t) \\ \hat{y}(t) = \hat{C}x(t) + \hat{D}u(t) \end{cases} \tag{9}$$

such as, $\hat{A} \in \mathbb{R}^{k\times k}$,$\hat{B} \in \mathbb{R}^{k\times p}$,$\hat{C} \in \mathbb{R}^{p\times k}$.

2.2 Description of SVD-AORA Model Order Reduction Method

The SVD-AORA method is a joint method which benefits from both Krylov and Singular value decomposition technique. This method generates two projection matrices V and Z. The first projection matrix V is generated by using of the AORA technique. The second matrix is computed by the use of the SVD technique and the matrix V is given according to this relation $Z = g_0 V(V^T g_0 V)^{-1}$, where g_o presents the Gramian observability matrix. The details of the SVD-AORA algorithm can be found in table 1 [21, 20, 6]:

Theorem 1 summarizes the principle of the proposed method .

Theorem 1: *Let a linear system as in 1 of order n and k expansion frequency* $(k << n)$. *Use the AORA algorithm to compute a first projection matrix V after a first k steps and the Lyaponuv technique to generate the observability Gramian matrix* g_o. *Then the second projection matrix Z is generated by the use of this relation:*

$$Z = WV(V^T WV)^{-1} \tag{10}$$

where the projection matrix W is a diagonal matrix, containing in the diagonal the first k singular values determined from the SVD decomposition of Gramian matrix.

2.3 Application

To test the SVD-AORA algorithm, we take two SISO models of different order (Eady of order 598 [22, 23] and RLC model of order 150 [24]). We present of each model the frequency response of the original model and the reduced one, the absolute error between the original model and reduced one and the poles distribution of the reduced model.

2.3.1 Model 1: Eady 598

The Eady model presents a mathematical model of atmospheric storm track (for example the region in the mid-latitude Pacific [22]). Its a SISO dynamical linear system of order 598 [22].

The figure 1 presents the frequency response of original system (Exact-598) and reduced one (SVD-AORA-16) of order 16. We notice a good correlation between the original and reduce one about the frequency range.

The absolute error variation between the original system and reduced one is shown in figure 2. We notice also from this result that there exist a good correlation between the original and reduced one.

The figure 3 depicts the poles distribution of the reduced system of order 16, we note that all poles are negative real part, which explain the preservation of stability.

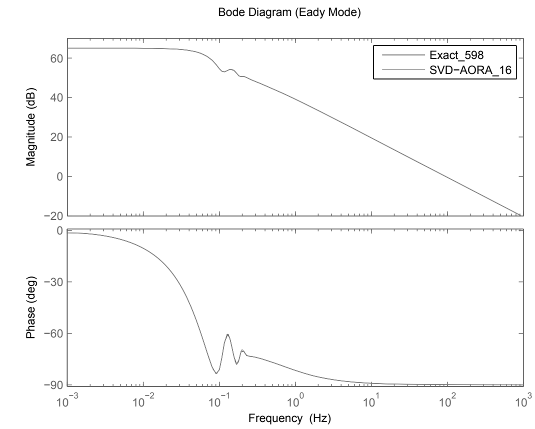

Figure 1: Frequency response of original system (Exact-598) and reduced one (SVD-AORA-16).

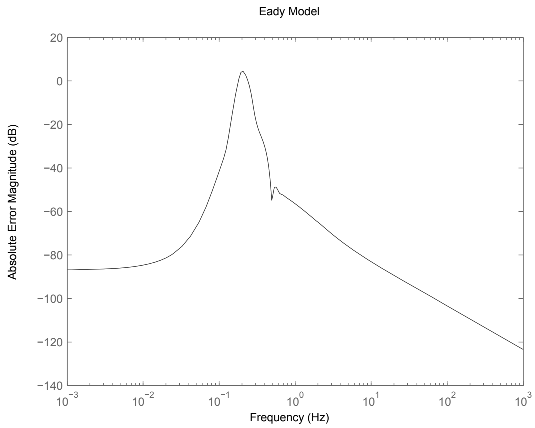

Figure 2: Absolute error between original model (598) and reduced one (16).

2.3.2 Model 2: N-RLC 150

The RLC model is a dynamical system with single-input/single-output [25, 26, 27, 11], it is very met in modeling of the electrical and electronic systems. The N-RLC model contains N chain of RLC circuit (where $R_N = 10k\Omega, C_N = 680\mu F$ and $L = 0.1H$ for $N = 1:50$). The electronic schematic of our N-RLC network is presented in figure 4:

The figure 5 depicts the frequency response of original system (Exact-150) and reduced one (SVD-AORA-12). We notice a good correlation between the original system and reduced one.

The figure 6 presents the absolute error variation between the original system and reduced one. We also

Table 1: SVD-AORA Model Order Reduction algorithm.

SVD-AORA Model Order Reduction algorithm:(Inputs:$A;B;C;D;S;k$**; Outputs:**$V;Z$**)**

(1): Define a frequency range S
$S = [s_1, s_2, ..., s_k]$ (with k $\ll$ n)
(2): Define a matrix ψ and a vector ξ for each expansion frequency s_i:
$\psi_i = -(s_i E - A)^{-1} E$ for i=1:k
$\xi_i = (s_i E - A)^{-1} B$ for i=1:k
(3): Compute the first projection matrix V using the AORA algorithm
$V = span\{\xi_1, \psi_2 \xi_2, ..., \psi_k^{k-1} \xi_2\}$
(4): Compute the gramian observability matrix g_o by solving the following Lyaponuv equation:
$A^T g_o + g_o A + C^T C = 0$
(5):Compute the singular value of the g_o matrix
$[U, W, T] = SVD(g_o)$
(6): Compute the second projection matrix Z through the following relation:
$Z = WV(V^T WV)^{-1}$
(7):The reduced system parameters can be defined by the congruences transformation
$\hat{E} = Z^T EV, \hat{A} = Z^T AV, \hat{B} = Z^T B, \hat{C} = C^T V$

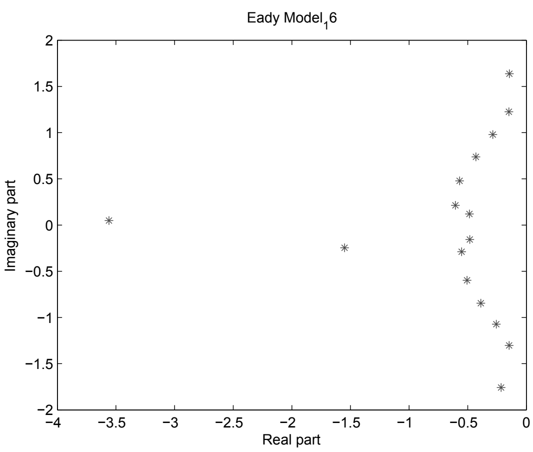

Figure 3: Poles Distribution of Eady reduced model with SVD-AORA method.

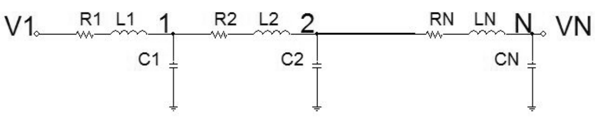

Figure 4: Chain RLC.

notice from the figure a good correlation between the original system and reduced one.

The poles distribution is depicted in the figure 7. We see that the all poles are negative real part, which explain the stability preservation of reduced system.

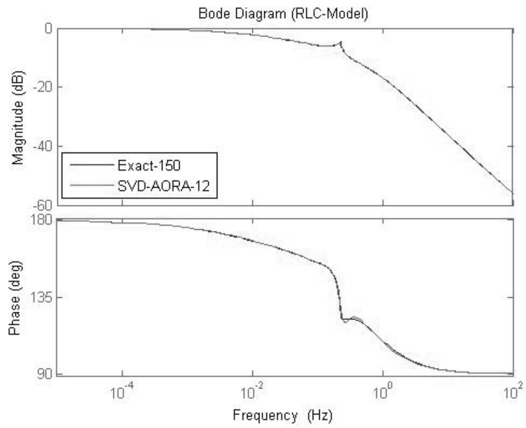

Figure 5: Frequency response of original system (Exact-150) and reduced one (SVD-AORA-12).

3 COMPARATIVE STUDY

In this section we present a comparative study between the SVD-AORA method and the SVD-Krylov one. Firstly, we present the frequency responses and the absolute error variations obtained by the tow methods. We depict also the poles distribution obtained by the SVD-Krylov method. Secondly, we give a comparative table containing the CPU-Time and the H_∞norm error for each method.

Figure 8 presents the frequency response of original system (order 598) and reduced one (order 16) obtained

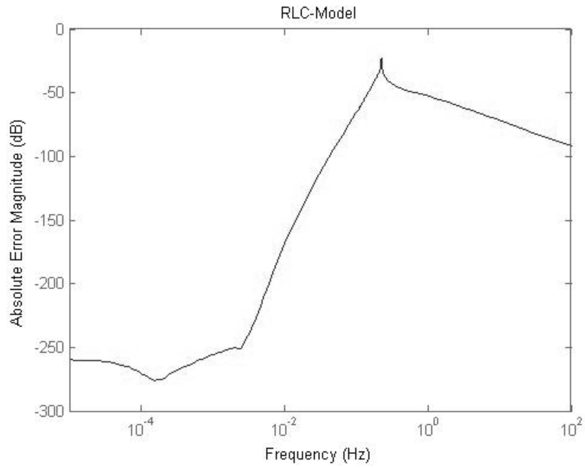

Figure 6: Absolute error between original system (150) and reduced one (12).

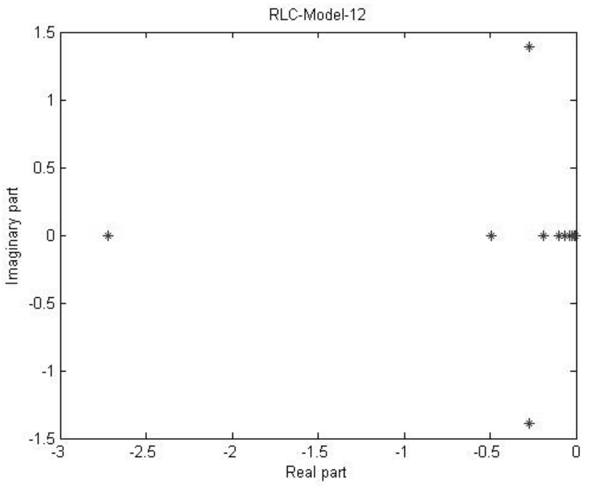

Figure 7: Poles Distribution of RLC reduced model (12) with SVD-AORA method.

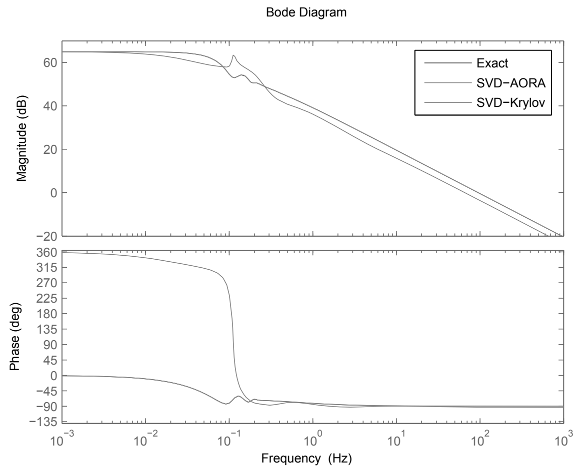

Figure 8: Frequency response of original system (Exact-598) and reduced one (16) with two methods.

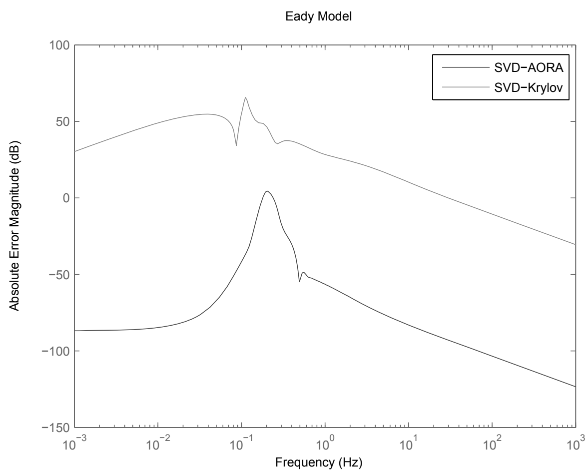

Figure 9: Absolute error between original model (598) and reduced one (16).

by the two methods. We notice a good correlation between the original system and reduced one for the result obtained by the SVD-AORA method.

We notice also from the figure 9 of the absolute error variation that the best result is obtained by the SVD-AORA method.

We note from the figure 10 of poles distribution obtained by the SVD-Krylov method the existence of positive real part poles, which explain the instability of reduced system.

Figure 11 shows the frequency response of original system (RLC-150) and reduced one (order 12) obtained by the two methods (SVD-AORA and SVD-Krylov). We note that the result obtained by the SVD-AORA method is very close to the original system which is not the case for the SVD-Krylov method.

Figure 12 shows the variation of absolute error between the original system and reduced one obtained according to the previous frequencies responses. The variation error between the original system and reduced one is very small near the low frequency and relatively small near the high frequency by the SVD-AORA method which is not the case for the SVD-Krylov method.

We note from the figure 13 of poles distribution ob-

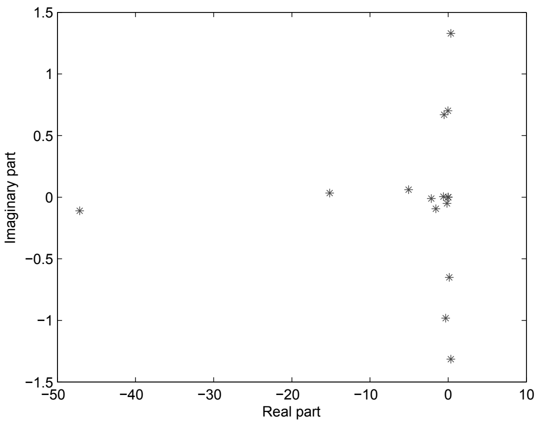

Figure 10: Poles Distribution of Eady reduced model (16) with SVD-Krylov method.

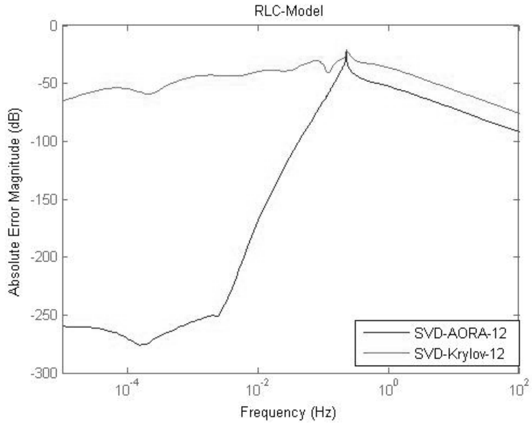

Figure 12: Absolute error between original model (150) and reduced one (12) with two methods (SVD-AORA-12 and SVD-Krylov-12).

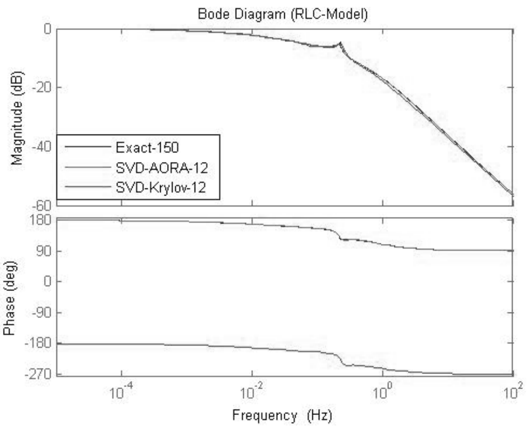

Figure 11: Frequency response of original system (Exact-150) and reduced one with two methods (SVD-AORA-12 and SVD-Krylov-12).

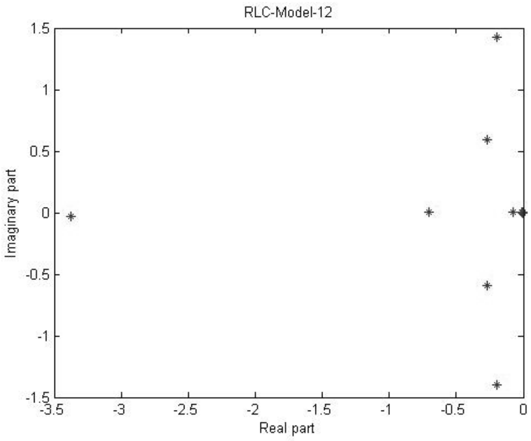

Figure 13: Poles Distribution of RLC reduced model (12) with SVD-Krylov method.

tained by the SVD-Krylov method that all the poles are negative real part, then the reduced system is stable. The table 2 contains the different values of H_∞ norm error and CPU-Time of each method. We note from the figures 8, 9, 11, 12 and from the table 2 that the best performance is obtained by the proposed method SVD-AORA.

4 Conclusion

A combined SVD-AORA method for dynamic linear time invariant model order reduction have been presented. The proposed method combine two techniques, which are the singular value decomposition and the Krylov. The Krylov technique is used in generation of first projection matrix, which is numerically efficiency. The singular value decomposition is used in computing the second projection matrix by the using of the Layponuv technique and the first projection matrix. Two models of different order were provided to prove why model order reduction via a combined techniques (SVD and Krylov) has the potential for significants improvement over existing combined method.

Table 2: H_∞norms and CPU-Time of each method.

Methods	SISO LTI System	minH_∞	maxH_∞	CPU-Time
SVD-AORA	Eady598	1.51110^{-7}	2.76010^{-5}	111.311s
SVD-Krylov	Eady598	0.044	6.693	98.755s
SVD-AORA	RLC150	5.88410^{-15}	4.34310^{-4}	13.023s
SVD-Krylov	RLC150	1.758910^{-4}	0.6290	11.823s

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